From toric varieties to Fano manifold

Thomas Prince

The study of toric geometry represents a point of contact between two very different fields. On the one hand toric varieties are objects in algebraic geometry, a special class of rational varieties; while on the other hand their geometry can be studied through the combinatorics of certain polyhedral objects one can associate with these varieties. We review the basic machinery of this theory, looking at toric varieties as algebraic varieties (or schemes), via their fans, as projective complex manifolds via polytopes, and as quotient spaces via GIT.

Toric geometry plays an important role in the field of mirror symmetry which studies a deep duality between Calabi–Yau manifolds or, more generally, Landau–Ginzburg models. Indeed mirror symmetry can often be understood in this context as a duality between the combinatorial objects which appear in toric geometry, making the subject particularly explicit.

Focussing on the case of Fano manifolds, we outline two possible (related) roles of mirror symmetry and toric geometry in the study of Fano manifolds. The first makes use of toric varieties as ambient spaces for more general varieties; for example as complete intersections, or degeneracy loci of maps between vector bundles. In particular we describe how mirror symmetry can be used to determine Gromov–Witten invariants of Fano manifolds embedded in certain toric varieties, particularly following work of Coates–Cort–Galki–Kaspzryk on quantum periods of Fano threefolds. The second centres on the notion that Fano manifolds can be related to toric varieties by the degeneration of a Fano manifold to a (possibly singular) toric variety. In particular we indicate how mirror symmetry offers the possibility of a 'dual' classification problem, and suggests– at least experimentally – methods to construct new Fano manifolds via toric degeneration. Finally, we survey joint work with Coates and Kasprzyk in which we use these ideas from mirror symmetry to identify four dimensional Fano manifolds, and in particular construct 738 four dimensional Fano manifolds as toric complete intersections. In this work we make use of the classification of smooth toric varieties in dimensions up to eight, and search for all Fano complete intersections in these varieties which meet certain natural conditions. We distinguish these Fano manifolds using their 'quantum period' which is itself calculated using mirror symmetry.