Algebraic varieties and their symmetries

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An algebraic variety is a set of common solutions of a system of polynomial equations. Except for the usual concept of isomorphism between two algebraic varieties, there is also a notion of a birational isomorphism, allowing to figure out which systems of equations are "essentially equivalent". Studying the groups of automorphisms and birational automorphisms is important for understanding the general structure of algebraic varieties and their classification.

While automorphism groups of algebraic varieties usually have a relatively nice structure, their groups of birational automorphisms are much more complicated. For instance, the group of birational selfmaps of a projective plane over the field of complex numbers (known as the Cremona group) is infinite-dimensional, and does not have any nontrivial finite-dimensional linear representations. However, such groups become more accessible if we study them on the level of their finite subgroups.

The main goal of my lectures is to survey a rough classification of algebraic varieties of dimensions 1 and 2 (that is, of curves and surfaces), and to describe an approach to the study of finite groups acting by their birational selfmaps. In the surface case, both of these are based on the Minimal Model Program, which was established by the classical Italian school of algebraic geometry (including Castelnuovo, Enriques, and others) and then reviewed and generalized into higher dimensions by Mori in the late 20th century. This is an algorithm allowing to "improve" the properties of a given surface staying in the same class of birational equivalence. Starting from any smooth projective surface, it produces its minimal model, having nice properties formulated in terms of intersection theory of curves on a surface and behavior of families of rational curves on it. Going a little bit further in this direction one can obtain a classical Kodaira-Enriques classification of projective (or, more generally, compact complex) surfaces. Moreover, using an equivariant version of the Minimal Model Program with respect to a finite group acting on a surface, one can reduce the questions about the latter groups to similar questions about groups acting on minimal models. This allows to obtain various explicit classification results for finite groups of automorphisms and birational automorphisms of algebraic surfaces, as well as certain general qualitative theorems concerning such groups.