Flag varieties, Schubert calculus and Gelfand–Zetlin polytopes

Evgeny Smirnov

The goal of these lectures is to provide an introduction into Schubert calculus on Grassmannians and flag varieties. A (rather optimistic) plan is as follows.

The first lecture will be devoted to Grassmannians. We will show that they are projective algebraic varieties and define their particularly nice cellular decomposition: the Schubert decomposition. We show that the cells of this decomposition are indexed by Young diagrams, and the inclusion between their closures, Schubert varieties, is also easily described in this language. Then we pass to the cohomology rings of Grassmannians and state the Pieri rule, which allows us to multiply cycles in the cohomology ring of a Grassmannian by a cycle of some special form. Time permitting, we will establish a relationship of this theory with the theory of symmetric functions. Finally, we will see how Schubert calculus can be used for solving problems of enumerative geometry.

The second lecture will be about full flag varieties. We mostly follow the same pattern: we define their Schubert decomposition, describe the inclusion order on the closures of Schubert cells, describe the structure of the coholomology ring of a full flag variety and formulate the Monk rule for multiplying a Schubert cycle by a divisor. Then we define a presentation of Schubert cycles by the so-called Schubert polynomials, and discuss the related combinatorics. We are going to conclude with the results of Allen Knutson and Ezra Miller on Gröbner toric degenerations of matrix Schubert varieties.

In our last lecture I will present an approach to Schubert calculus using Gelfand–Zetlin polytopes. This approach allows us to compute the intersection products of Schubert cycles by intersecting faces of a polytope (this is our joint result with V.Kiritchenko and V.Timorin).