Approximate Bandgap Computation Using Fourier Series for One-Dimensional Isotropic Dielectric-Magnetic Photonic Crystals

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2016 – 12 – 0006

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MS Thesis
MAY 2018

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The one-dimensional isotropic dielectric-magnetic photonic crystals are materials with periodically varying permittivity and periodically varying permeability along the same direction. These photonic crystals offer more degrees of freedom in designing photonic crystals. Furthermore, these photonic crystals find applications in sub-wavelength imaging and diffraction-less propagation. Using the coupled-mode analysis based on Fourier series, we have worked out the band-edge frequencies of the one-dimensional isotropic dielectric-magnetic photonic crystals for the axial and oblique propagation. We also worked on the wave number for the propagating and evanescent wave. Numerical results for a photonic crystal with six layers in each unit cell were also computed and compared with exact results.
Dedication and Acknowledgements

First of all, I would like to thank Allah Almighty for giving the strength to me to do this thesis. Then I want to say that this work would not be possible without the guidance of my supervisor, support of my family and help from my friends. I am very much thankful for my supervisor Dr. Muhammad Faryad, for his guidance, encouragement and support which provided me an opportunity to complete my research work under his supervision.

Moreover, I am thankful to all of my friends at LUMS, Lahore, especially Ayesha Noureen, Hassan Wasalat, Yasir Iqbal, Wajiha Haseeb, Shahid Tanveer, Hafiz Arslan Hashim, Junaid Saif Khan, Asif Nawaz, Syed Waqar Ahmed, Muhammad Kamran, Noman Safdar, Subhan Jamil and Amir Hayat for all kind of their support. I would also like to thank the staff of my department especially Mr. Arshad Maral for his support.

Finally, I would like to gratefully acknowledge the support and encouragement of my parents throughout my studies.
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This chapter gives a short introduction to the basics of dielectric-magnetic photonic crystals. These photonic crystals have periodically varying permittivity and permeability. These photonic crystals are made up of dielectric, metal-dielectric or some nanostructures that will affect the electromagnetic wave propagation as in the case of periodic potential in a semiconducting crystals which affects the electron’s motion by defined allowed and forbidden energy band gaps. We generally looking for the fact that what these crystals are and then describe the different type of photonic crystals. Our main focus is to study the band structure and photonic band gaps.

1.1 Photonic Crystals

The optical analogue of the ionic crystals are the photonic crystals, in these crystals the atoms or molecules are dielectric constants with the replacement of macroscopic media and the periodic potentials are replaced by the dielectric functions which are periodic. The reflections and refraction of light from the various places or interfaces which can produce many of the phenomena for the light modes, where these light modes are called photons, these reflections and refractions are done when the dielectric constant of the
given materials are slightly different and the absorption of light by the material is small. The photonic crystals are loss-less periodically repeated dielectric structures. These structures can be made by the photonic band gap which are the property of disallowing the propagation of the light. Metallic wave guides and certain cavities will be used for controlling the microwave propagation [1] [2].

![Image of Photonic Crystals]

Figure 1.1: One, Two and Three Dimensional Photonic Crystals

The photonic crystals are explained by the three main types which are one, two and three dimensional photonic crystals. In 1D crystals, the layers of such type of photonic crystals having different dielectric constant but they form a band gap in the single direction. The Bragg grating is one of the example from one dimensional photonic crystals. In 2D crystals, these are crystals which having a change in dielectric constant in two of the axes and the third one is uniform. These 2D crystals can be made by drilling the holes in the given substrate. Finally, the 3D crystals having the change in the dielectric constant in all the three axes. There fabrication is done by placing the 2D layers on the top of the every layer and a 3D structure is obtained. The first 3D structure for the photonic crystal obtained was Yablonovite which also named as spheres in a diamond lattice. [3]

1.2 Magnetic Photonic Crystals

Recently, the interest in studying or designing the photonic crystals is going very deep and they take into account another type of photonic crystal which having periodically
varying permeability those photonic crystals are called magnetic photonic crystals or magnetophotonic crystals. Magnetic photonic crystals have been an important part in studying theoretically and experimentally over the past several years. These type of crystal structures having some important and interesting properties such as Enhanced Faraday Effect [4], magnetic super-prism effect [5] and nonreciprocal or magnetically controllable photonic [6],[7], these all properties are predicted theoretically.

Among all of the properties of the photonic crystals, the property which is very much important for the study of photonic crystals is the formation of band gap. In studying the magnetic photonic crystals many of the properties of these structures are studied or analyzed. These properties are the band gap formation, local normal mode coupling, Bloch states in birefringent magneto photonic periodic stacks [8] and degenerate band gap periodic magneto-optic systems [9]. The magnetic photonic crystal can also defined for the materials whose permeability is one but they have relative permittivity tensor because this tensor is dependent on the externally applied low-frequency magnetic field [10].

1.3 Metallic Photonic Crystals

These photonic crystals are formed by the periodic structures of defined and excess of resonant cavities which selectively absorbs the incident radiations on the crystals. Inside the resonant cavities, the metal layer is fixed and before that the dielectric materials are filled inside the cavities. These photonic crystals have gained the attention for the designing or studying due to there strong non-local response in the effective medium regime, in this regime the thicknesses of these layers are very much smaller than the the wavelength of the system [11].

These crystals can also give the information about the effective chiral theory (in QCD, at low energies it describes the process of strong interaction) and also about the epsilon-near-zero material (these materials found in nature like electron gas). These photonic crystals having more applications in the subwavelength imaging [12] and diffractionless
propagation [13]. These photonic crystals can also be used as ultrasensitive molecular detectors to down the concentration to some picomolar level and this all is done under surface-enhanced Raman spectroscopy. These crystals also find there applications in the many more other areas such as plasmonic devices and nanophotonic crystals.
In the previous chapter, the understanding about the photonic crystals is explained, what are these crystals? and how many ways to explain them? In this chapter, we will discuss the one dimensional isotropic dielectric magnetic photonic crystals and we will work out in calculating the spectral band edges, band central frequencies and also the calculation of the wave number for the propagating and the evanescent wave for the TE and TM polarization mode. These all calculations will be done by using the Fourier Series method, in which the dielectric parameters will be treated as a Fourier Series. At the next chapter we will show the conclusion and some numerical results.

### 2.1 Axial Propagation:

#### 2.1.1 Band-Edge Frequencies

The axial or normal propagation as shown by the given Fig. (2.1). The Maxwell equation can be written as:

\[ \nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}. \]  

(2.1)
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Figure 2.1: Wave propagating along z-axis (Axial Propagation)

Now using;

\[ \mathbf{B} = \mu \mathbf{H}. \]  \hfill (2.2)

We get

\[ \frac{1}{\mu} (\nabla \times \mathbf{E}) = -\frac{\partial \mathbf{H}}{\partial t} \]  \hfill (2.3)

Taking curl of the above equation then we get

\[ \nabla \times \frac{1}{\mu} (\nabla \times \mathbf{E}) = -\frac{\partial}{\partial t}(\nabla \times \mathbf{H}) \]  \hfill (2.4)

The wave equation for the one dimensional periodic layered media can be written as:

\[ \nabla \times \frac{1}{\mu} (\nabla \times \mathbf{E}) + \frac{\partial}{\partial t}(\nabla \times \mathbf{H}) = 0, \]  \hfill (2.5)

Using

\[ \nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t} \]  \hfill (2.6)

\[ \nabla \times \frac{1}{\mu} (\nabla \times \mathbf{E}) + \epsilon \frac{\partial^2 \mathbf{E}}{\partial t^2} = 0 \]  \hfill (2.7)

The permittivity (\(\epsilon\)) and permeability (\(\mu\)) can be written in a Fourier series as:

\[ \epsilon(z) = \sum_n \epsilon_n e^{-i n g z}, \]  \hfill (2.8)

\[ \frac{1}{\mu(z)} = \sum_m \mu_m e^{-i m g z}, \]  \hfill (2.9)

where

\[ g = \frac{2\pi}{\Lambda}. \]  \hfill (2.10)
Equation (2.7) for time harmonic fields can be written as:

$$\nabla \times \frac{1}{\mu} (\nabla \times \mathbf{E}) - \omega^2 \epsilon \mathbf{E} = 0$$  \hspace{1cm} (2.11)

In general, the electric field in this periodic medium may be expressed as a Fourier integral

$$\mathbf{E} = \int_{-\infty}^{\infty} dk'_z A(k'_z) e^{-i k'_z z}$$ \hspace{1cm} (2.12)

Now putting the value of electric field in the Eq. (2.11) and writing this wave equation in term of $k$ we get:

$$\int_{-\infty}^{\infty} dk'_z k'_z \hat{z} \times \frac{1}{\mu(z)} k'_z \hat{z} \times A(k'_z) e^{-i k'_z z} + \omega^2 \epsilon(z) \int_{-\infty}^{\infty} dk'_z A(k'_z) e^{-i k'_z z} = 0$$ \hspace{1cm} (2.13)

After putting the values of $\epsilon(z)$ and $\frac{1}{\mu(z)}$ from Eqs. (2.8) and (2.9) in Eq. (2.13) we get:

$$\int_{-\infty}^{\infty} dk'_z k'_z \hat{z} \times \left( \sum_m \bar{\mu}_m e^{-i m g z} \right) k'_z \hat{z} \times A(k'_z) e^{-i k'_z z} + \omega^2 \sum_n \epsilon_n e^{-i n g z} \int_{-\infty}^{\infty} dk'_z A(k'_z) e^{-i k'_z z} = 0$$ \hspace{1cm} (2.14)

or

$$\int_{-\infty}^{\infty} dk'_z k'_z \hat{z} \times \left( \sum_m \bar{\mu}_m e^{-i m g z} \right) k'_z \hat{z} \times A(k'_z) e^{-i k'_z z} + \omega^2 \sum_n \epsilon_n \int_{-\infty}^{\infty} dk'_z A(k'_z) e^{-i (k'_z + n g) z} = 0$$ \hspace{1cm} (2.15)

Assuming the field $\mathbf{E}$ to be transverse to the direction of propagation i.e., $\hat{z} \cdot \mathbf{A} = 0$ we get

$$\sum_m \bar{\mu}_m \int_{-\infty}^{\infty} dk'_z k'^2 A(k'_z) e^{-i (k'_z + m g) z} - \omega^2 \sum_n \epsilon_n \int_{-\infty}^{\infty} dk'_z A(k'_z) e^{-i (k'_z + n g) z} = 0$$ \hspace{1cm} (2.16)

which can be written as

$$\int_{-\infty}^{\infty} dk''_z \left[ \sum_m \bar{\mu}_m (k''_z - m g)^2 A(k''_z - m g) - \omega^2 \sum_n \epsilon_n A(k''_z - n g) \right] e^{-i k''_z z} = 0$$ \hspace{1cm} (2.17)

The above equation is valid only when all the coefficients of $e^{-i k''_z z}$ vanish. Thus

$$\sum_m \bar{\mu}_m (k''_z - m g)^2 A(k''_z - m g) - \omega^2 \sum_n \epsilon_n A(k''_z - n g) = 0$$ \hspace{1cm} (2.18)
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To find the Bloch wave number $K$, we have to solve the set of Eqs. (2.18) with $k''_z = K, K \pm g, K \pm 2g, \ldots$. By doing this we get some equations in term of $A(K), A(K \pm g)$... and so on for this we apply some approximations.

We get

\[ 0 = \ldots \mu_1(K+g)^2 A(K+g) + \bar{\mu}_0 K^2 A(K) + \bar{\mu}_1(K-g)^2 A(K-g) + \ldots \]

(2.19)

By rearranging the Eq. (2.19) for every coefficient of $A$ we get the following equation;

\[ A(K) = \frac{1}{\bar{\mu}_0 K^2 - \omega^2 \varepsilon_0} \left\{ \left[ \omega^2 \varepsilon_1 - \bar{\mu}_1 (K+g)^2 \right] A(K+g) + \left[ \omega^2 \varepsilon_1 - \bar{\mu}_1 (K-g)^2 \right] A(K-g) + \ldots \right\} \]

(2.20)

Similarly, for $k''_z = K - g$ in Eq. (2.18) we get

\[ A(K-g) = \frac{1}{\bar{\mu}_0 (K-g)^2 - \omega^2 \varepsilon_0} \left\{ \left[ \omega^2 \varepsilon_1 - \bar{\mu}_1 (K-2g)^2 \right] A(K-2g) + \left[ \omega^2 \varepsilon_1 - \bar{\mu}_1 (K+g)^2 \right] A(K+g) + \ldots \right\} \]

(2.21)

and for $k''_z = K + g$

\[ A(K+g) = \frac{1}{\bar{\mu}_0 (K+g)^2 - \omega^2 \varepsilon_0} \left\{ \left[ \omega^2 \varepsilon_1 - \bar{\mu}_1 K^2 \right] A(K) + \left[ \omega^2 \varepsilon_1 - \bar{\mu}_1 (K+2g)^2 \right] A(K+2g) + \ldots \right\} \]

(2.22)

Now, if we inspect Eqs. (2.20), (2.21) and (2.22) we can see that

\[ |K - g| = K \]

(2.23)

and

\[ K^2 = \frac{\omega^2 \varepsilon_0}{\bar{\mu}_0} \]

(2.24)

Implies that terms of $A(K)$ and $A(K - g)$ are dominant and we can neglect all the other terms. This reduces Eqs. (2.20) and (2.21) to

\[ [\bar{\mu}_0 K^2 - \omega^2 \varepsilon_0] A(K) - [\omega^2 \varepsilon_1 - \bar{\mu}_1 (K-g)^2] A(K-g) = 0 \]

(2.25)

\[ - [\omega^2 \varepsilon_1 - \bar{\mu}_1 (K+2g)^2] A(K) + [\bar{\mu}_0 (K-g)^2 - \omega^2 \varepsilon_0] A(K-g) = 0 \]

(2.26)
Which have non-trivial solution if

\[
\begin{vmatrix}
\mu_0 K^2 - \omega^2 \epsilon_0 & \mu_1 (K - g)^2 - \omega^2 \epsilon_1 \\
\mu_{-1} K^2 - \omega^2 \epsilon_{-1} & \mu_0 (K - g)^2 - \omega^2 \epsilon_0
\end{vmatrix} = 0
\] (2.27)

Using Eq. (2.23)

\[
(\mu_0 K^2 - \omega^2 \epsilon_0)^2 - (\mu_1 K^2 - \omega^2 \epsilon_1)(\mu_{-1} K^2 - \omega^2 \epsilon_{-1}) = 0
\] (2.28)

Which can be solved to get

\[
\omega^2 = \frac{K^2 \left( (\epsilon_{-1} \mu_1 + \epsilon_1 \mu_{-1} - 2\mu_0 \epsilon_0) \pm \sqrt{(2\mu_0 \epsilon_0 - \epsilon_{-1} \mu_1 - \epsilon_1 \mu_{-1})^2 - 4(\epsilon_1^2 - \epsilon_0^2)(\mu_1^2 - \mu_{-1}^2)} \right)}{2(\epsilon_1^2 - \epsilon_0^2)}
\] (2.29)

Assuming

\[
\epsilon_{-1} \mu_1 + \epsilon_1 \mu_{-1} \approx 2|\epsilon_1||\mu_1|
\] (2.30)

and neglecting terms like \(\epsilon_{-1} \mu_1^2\) and \(\epsilon_1 \mu_{-1}^2\) because they are small Eq. (2.28) becomes

\[
\omega^2 = \frac{K^2 \mu_0 \mp |\mu_1|}{\epsilon_0 \mp |\epsilon_1|}
\] (2.31)

These are the spectral band edges. At the frequencies \(\omega\) between \(\omega_+\) and \(\omega_-\), the roots of the Eq. (2.27) are complex, with real part equal to \(\frac{\pi}{\Lambda}\) and the waves are evanescent. Between these frequencies the band is forbidden band gap and outside this band gap, the roots of this equations are real for \(K\) and the wave is propagating wave.

This is the result for first order band, for the higher order bands the Fourier coefficients of \(\mu\) and \(\epsilon\) are going to be increased. So for the general result we have following:

\[
\omega^2 = \frac{K^2 \mu_0 \mp |\mu_l|}{\epsilon_0 \mp |\epsilon_l|}
\] (2.32)

where \(l = \pm 1, \pm 2, \ldots\)

The central frequency between \(\omega_+\) and \(\omega_-\) can be written as:

\[
\omega_0 = \frac{\omega_+ + \omega_-}{2}
\] (2.33)

By putting the values of \(\omega_+\) and \(\omega_-\) we get the following:

\[
\omega_0 = \frac{K}{2} \sqrt{\frac{\mu_0}{\epsilon_0}} \sqrt{1 - |\mu_1| \mu_0} \sqrt{1 + |\epsilon_1| \epsilon_0} + \sqrt{1 + |\mu_1| \mu_0} \sqrt{1 - |\epsilon_1| \epsilon_0}
\] (2.34)
Now solving the above equation and applying the binomial theorem we get our final result:

\[
\omega_0 = K \sqrt{\frac{\mu_0}{\varepsilon_0}} \left( 1 - \frac{|\mu_1| \varepsilon_1}{4\mu_0 \varepsilon_0} \right)
\]  

(2.35)

and the band gap can also be defined by the formula as:

\[
\Delta\omega_{gap} = |\omega_+ - \omega_-|
\]

(2.36)

By putting the values in the above equation of \(\omega_+\) and \(\omega_-\) we get the following equation:

\[
\Delta\omega_{gap} = K \left( \frac{\sqrt{\mu_0 - |\mu_1|}}{\sqrt{\varepsilon_0 - |\varepsilon_1|}} - \frac{\sqrt{\mu_0 + |\mu_1|}}{\sqrt{\varepsilon_0 + |\varepsilon_1|}} \right)
\]

(2.37)

The final result can be written as:

\[
\Delta\omega_{gap} = K \sqrt{\frac{\mu_0}{\varepsilon_0}} \left( \frac{|\varepsilon_1|}{\varepsilon_0} - \frac{|\mu_1|}{\mu_0} \right)
\]

(2.38)

These are our required result for the spectral band edges this gives us a forbidden band gap between the \(\omega_+\) and \(\omega_-\) because in this region the solution for the \(K\) is complex and the wave is evanescent. This relation is known as dispersion relation.

### 2.1.2 Wave Number in Band Gap

Now, we want to find out the imaginary value of wave number \(K\) in the band gap. Eq. (2.27) gives

\[
[\bar{\mu}_0 K^2 - \omega^2 \varepsilon_0] [\bar{\mu}_0 (K - g)^2 - \omega^2 \varepsilon_0] - [\omega^2 \varepsilon_1 - \bar{\mu}_1 (K - g)^2] [\omega^2 \varepsilon_{-1} - \bar{\mu}_{-1} K^2] = 0
\]

(2.39)

We know that

\[
g = \frac{2\pi}{\Lambda}
\]

(2.40)

and also the value of \(K\) can be written as:

\[
K = \frac{\pi}{\Lambda} + x
\]

(2.41)

By putting the values of \(g\) and \(K\) in Eq. 2.39, we get the following equation:

\[
0 = \left[ \bar{\mu}_0 \left( \frac{\pi}{\Lambda} + x \right)^2 - \omega^2 \varepsilon_0 \right] \left[ \bar{\mu}_0 \left( x - \frac{\pi}{\Lambda} \right)^2 - \omega^2 \varepsilon_0 \right] - \left[ \omega^2 \varepsilon_1 - \bar{\mu}_1 \left( x - \frac{\pi}{\Lambda} \right)^2 \right] \left[ \omega^2 \varepsilon_{-1} - \bar{\mu}_{-1} \left( x + \frac{\pi}{\Lambda} \right)^2 \right]
\]

(2.42)
Now, in the above equation opening the squares and all the other terms we get:

\[
0 = \bar{\mu}_0 \left[ \left( (x + \frac{\pi}{\Lambda})^2 - \frac{\omega^2 \varepsilon_0}{\bar{\mu}_0} \right) \left( (x - \frac{\pi}{\Lambda})^2 - \frac{\omega^2 \varepsilon_0}{\bar{\mu}_0} \right) \right] \\
- \left[ \frac{\omega^2 \varepsilon_1 - \bar{\mu}_1}{\Lambda^2} \left( x^2 + \frac{\pi^2}{\Lambda^2} - \frac{2\pi x}{\Lambda} \right) \right] \left[ \frac{\omega^2 \varepsilon_{-1} - \bar{\mu}_{-1}}{\Lambda^2} \left( x^2 + \frac{\pi^2}{\Lambda^2} + \frac{2\pi x}{\Lambda} \right) \right]
\]

and neglecting all the \(x^4\) terms, we get:

\[
0 = \bar{\mu}_0 \left[ \left( \frac{\pi^2}{\Lambda^2} - \frac{\omega^2 \varepsilon_0}{\bar{\mu}_0} \right) \right] - \left[ \frac{2\pi^2}{\Lambda^2} + \frac{2\omega^2 \varepsilon_0}{\bar{\mu}_0} \right] x^2 \]

\[
- \left[ (\omega^2 |\varepsilon_1|)^2 - \left( 2\omega^2 |\varepsilon_1| \frac{|\bar{\mu}_1|}{\Lambda} \right) \right] x^2 + \frac{\pi^2}{\Lambda^2} \left( \frac{\pi^2 |\bar{\mu}_1|^2}{\Lambda^2} - \frac{2\omega^2 |\varepsilon_1| |\bar{\mu}_1|}{} \right)
\]

Solving for \(x^2\) we get

\[
x^2 = \frac{K^4 \left( \frac{|\bar{\mu}_1|^2}{\bar{\mu}_0} \right) - K^2 \left( 2\omega^2 |\varepsilon_1| |\bar{\mu}_1| \right) + \omega^4 |\varepsilon_1|^2}{2K^2 \left( \frac{|\bar{\mu}_1|^2}{\bar{\mu}_0} \right) - 4K^2 + 2\frac{\omega^2}{\bar{\mu}_0} (|\varepsilon_1| |\bar{\mu}_1|)}
\]

After simplification, we get

\[
x^2 = \frac{\omega^2 \varepsilon_0 |\bar{\mu}_1|^2 - 2\omega^2 |\bar{\mu}_0| |\varepsilon_1| |\bar{\mu}_1| + \frac{\bar{\mu}_0}{\omega^2 \varepsilon_0} (\omega^2 |\varepsilon_1|)^2}{2 |\bar{\mu}_1| - 4\bar{\mu}_0 + \left( \frac{\bar{\mu}_0}{\omega^2 \varepsilon_0} \right) 2\omega^2 |\varepsilon_1| |\bar{\mu}_1|}
\]

### 2.2 Oblique Propagation

#### 2.2.1 TE Case

In this mode the direction of electric field is transverse in the direction of propagation of wave but the magnetic field is normal to the direction of propagation of wave. Now, we have to find out the band frequencies in the same way as we worked for the axial propagation. The method for solving the wave equation from starting is worked for the TE case but there is a change in electric field which is now in y and z directions.
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Figure 2.2: Wave vector is propagating along $z$ and $y$-axis. For TE, the electric field is transverse to the direction of propagation. For TM, magnetic field is transverse to the direction of propagation.

In general, the electric field in this periodic medium may be expressed as a Fourier integral

$$E = \int_{-\infty}^{\infty} dk_z' dk_y' \hat{x} A_x'(k_y', k_z') e^{-ik_z'z} e^{-ik_y'y}$$  \hspace{1cm} (2.47)

$$0 = \int_{-\infty}^{\infty} dk_z' dk_y' [k_y' \hat{y} + k_z' \hat{z}] \times \frac{1}{\mu(z)} [k_y' \hat{y} + k_z' \hat{z}] \times \hat{x} A_x'(k_z') e^{-ik_z'z} e^{-ik_y'y}$$

$$+ \omega^2 e(z) \int_{-\infty}^{\infty} dk_z' dk_y' \hat{x} A_x'(k_z') e^{-ik_z'z} e^{-ik_y'y}$$  \hspace{1cm} (2.48)

$$0 = \int_{-\infty}^{\infty} dk_z' dk_y' [k_y' \hat{y} + k_z' \hat{z}] \times \left( \sum_m \hat{x} A_x'(k_z') e^{-img\hat{z}} \right) \left( [k_y' \hat{y} + k_z' \hat{z}] \times \hat{x} A_x'(k_z') e^{-ik_z'z} e^{-ik_y'y} \right)$$

$$+ \omega^2 \sum_n \epsilon_n e^{-ing} \int_{-\infty}^{\infty} dk_z' dk_y' \hat{x} A_x'(k_z') e^{-ik_z'z} e^{-ik_y'y}$$  \hspace{1cm} (2.49)

$$0 = \sum_m \overline{\mu}_m \int_{-\infty}^{\infty} [k'_{2z}^2 + k_{2y}'^2] e^{-img \hat{z}} e^{-i(mg+k_z')z} A_x'(k_z') dk_z' dk_y'$$

$$- \omega^2 \sum_m \epsilon_n \int_{-\infty}^{\infty} e^{-ik_y'y} e^{-in\hat{g}z} A_x'(k_z') dk_z' dk_y'$$  \hspace{1cm} (2.50)
Now, the above equation can be written as
\[
\int_{-\infty}^{\infty} \left\{ \sum_{m} \left[ (k''_{z} - mg)^{2} + k'^{2}_{y} \right] A_{x}(k''_{z} - mg) - \omega^{2} \sum_{n} \epsilon_{n} A_{x}(k''_{z} - ng) \right\} e^{-ik'_{y}y} e^{-ik''_{z}z} dk''_{z}dk'_{y} = 0
\]  
(2.51)

We get
\[
\sum_{m} \bar{\mu}_{m} A_{x}(k''_{z} - mg) \left[ (k''_{z} - mg)^{2} + k'^{2}_{y} \right] - \omega^{2} \sum_{n} \epsilon_{n} A_{x}(k''_{z} - ng) = 0
\]  
(2.52)

Here \(m, n = 0, \pm 1, \pm 2, \pm 3, \ldots\) and to find the Bloch wave number for TE mode, we have to solve the above equation with \(k''_{x} = K_{x}, K_{x} \pm g, K_{x} \pm 2g, \ldots\)

\[
0 = \ldots + \bar{\mu}_{-1} [(K_{x} + g)^{2} + K'^{2}_{y}] A_{x}(K_{x} + g) + \bar{\mu}_{0}(K^{2}_{x} + K'^{2}_{y}) A_{x}(K_{x}) + \bar{\mu}_{1} [(K_{x} - g)^{2} + K'^{2}_{y}] A_{x}(K_{x} - g) + \ldots
\]  
(2.53)

or
\[
A_{x}(K_{x}) = \frac{1}{\bar{\mu}_{0}(K^{2}_{x} + K'^{2}_{y}) - \omega^{2} \epsilon_{0}} \left[ (\omega^{2} \epsilon_{-1} - \bar{\mu}_{-1} [(K_{x} + g)^{2} + K'^{2}_{y}] A_{x}(K_{x} + g) + \ldots \right.
\]  
(2.54)

Similarly, by using \(k''_{x} = K_{x} - g\) Eq. (2.52) we get

\[
A_{x}(K_{x} - g) = \frac{1}{\bar{\mu}_{0}[(K_{x} - g)^{2} + K'^{2}_{y}] - \omega^{2} \epsilon_{0}} \left[ (\omega^{2} \epsilon_{1} - \bar{\mu}_{1} [(K_{x} - 2g)^{2} + K'^{2}_{y}] A_{x}(K_{x} - 2g) + \ldots \right.
\]  
(2.55)

and for \(k''_{x} = K_{x} + g\) is

\[
A_{x}(K_{x} + g) = \frac{1}{\bar{\mu}_{0}[(K_{x} + g)^{2} + K'^{2}_{y}] - \omega^{2} \epsilon_{0}} \left[ (\omega^{2} \epsilon_{-1} - \bar{\mu}_{-1} [(K_{x} + 2g)^{2} + K'^{2}_{y}] A_{x}(K_{x} + 2g) + \ldots \right.
\]  
(2.56)

if
\[
| K_{x} - g | = K_{x}
\]  
(2.57)

and
\[
K^{2}_{x} + K'^{2}_{y} = \frac{\omega^{2} \epsilon}{\bar{\mu}_{0}}
\]  
(2.58)
Implies that the terms $A_x(K_z)$ and $A_x(K_z - g)$ are dominant we can neglect all the other terms we get:

$$A_x(K_z)[\mu_0(K_z^2 + K_y^2) - \omega^2 \epsilon_0] - \{\omega^2 \epsilon_1 - \mu_1 [(K_z - g)^2 + K_y^2]\} A_x(K_z - g) = 0 \quad (2.59)$$

and

$$-A_x(K_z)\left[(\omega^2 \epsilon_{-1} - \mu_{-1}(K_z^2 + K_y^2)] + \{\mu_0 [(K_z - g)^2 + K_y^2] - \omega^2 \epsilon_0\} A_x(K_z - g) = 0 \quad (2.60)$$

The Eqs. (2.59) and (2.60) form a matrix which have a non-trivial solution if

$$\begin{vmatrix} \mu_0(K_z^2 + K_y^2) - \omega^2 \epsilon_0 & \mu_1 [(K_z - g)^2 + K_y^2] - \omega^2 \epsilon_1 \\ \mu_{-1}(K_z^2 + K_y^2) - \omega^2 \epsilon_{-1} & \mu_0 [(K_z - g)^2 + K_y^2] - \omega^2 \epsilon_0 \end{vmatrix} = 0 \quad (2.61)$$

After solving and applying the quadratic formula, we get the following equation:

$$\omega = \sqrt{K_z^2 + K_y^2} \left\{ \frac{\mu_1 \epsilon_{-1} + \epsilon_1 \mu_{-1} - 2 \epsilon_0 \mu_0 \pm \sqrt{(2 \epsilon_0 \mu_0 - \mu_1 \epsilon_{-1} - \epsilon_1 \mu_{-1}) - 4(|\epsilon|^2 - \epsilon_0)(|\mu_1|^2 - \mu_0)}}{2(|\epsilon|^2 - \epsilon_0)} \right\}$$

Which can be solved to get

$$\omega^2 = (K_z^2 + K_y^2) \left( \frac{\mu_0 \mp |\mu_1|}{\epsilon_0 \mp |\epsilon_1|} \right) \quad (2.62)$$

The general result for finding the higher order band coefficients of dielectric-magnetic photonic crystal of spectral band edges for TE mode is given as:

$$\omega^2 = (K_z^2 + K_y^2) \left( \frac{\mu_0 \mp |\mu_l|}{\epsilon_0 \mp |\epsilon_l|} \right) \quad (2.63)$$

For 1st order the value of $K_z$ is $\frac{\pi}{\Lambda}$ and for the $l$th order the value of $K_z$ is $l \frac{\pi}{\Lambda}$ where $l = \pm 1, \pm 2, \pm 3, \ldots$

Now we have to find out the central frequency of the band by using the general formula, which can be written as:

$$\omega_0 = \frac{\omega_+ + \omega_-}{2} \quad (2.65)$$

By putting the values of $\omega_+$ and $\omega_-$ in the above equation and using binomial theorem we get:

$$\omega_0 = \frac{\sqrt{\mu_0(K_z^2 + K_y^2)}}{\epsilon_0} \left( 1 - \frac{|\mu_1||\epsilon_1|}{4\mu_0 \epsilon_0} \right) \quad (2.66)$$
The above equation is central frequency of the band. Now, we have to find out the expression for the band gap so the formula for calculating the band gap is:

$$\Delta \omega_{\text{gap}} = |\omega_+ - \omega_-|$$ \hspace{1cm} (2.67)

Similarly, using the values of $\omega_+$ and $\omega_-$ in above equation then solving it, we get:

$$\Delta \omega_{\text{gap}} = \sqrt{\frac{\mu_0}{\epsilon_0} \left( K_z^2 + K_y^2 \right) \left( \frac{|\epsilon_1|}{\epsilon_0} - \frac{|\mu_1|}{\mu_0} \right)}$$ \hspace{1cm} (2.68)

### 2.2.2 TM Case:

In this mode, the magnetic field of the propagating wave is transverse to the direction of propagation of the wave.

For the transverse magnetic case, the Maxwell equation for the magnetic field will be used and can be written as:

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t}$$ \hspace{1cm} (2.69)

Using

$$\mathbf{D} = \epsilon \mathbf{E}$$ \hspace{1cm} (2.70)

Equation (2.69) becomes

$$\nabla \times \mathbf{H} = \epsilon \frac{\partial \mathbf{E}}{\partial t}$$ \hspace{1cm} (2.71)

For time-harmonic fields

$$\frac{1}{\epsilon} (\nabla \times \mathbf{H}) = -i \omega \mathbf{E}$$ \hspace{1cm} (2.72)

Taking curl of Eq. (2.72), we get

$$\nabla \times \frac{1}{\epsilon} (\nabla \times \mathbf{H}) = -i \omega (\nabla \times \mathbf{E})$$ \hspace{1cm} (2.73)

The curl of an electric field for time-harmonic fields is given by

$$\nabla \times \mathbf{E} = i \omega \mu \mathbf{H}$$ \hspace{1cm} (2.74)

Now using the Eq. (2.74) into (2.73), we get

$$\nabla \times \frac{1}{\epsilon} (\nabla \times \mathbf{H}) - \omega^2 \mu \mathbf{H} = 0$$ \hspace{1cm} (2.75)
Now the wave equation for the magnetic field is the same as the electric field Eq. (2.11). Thus, the whole calculation is same, here we use Fourier series for $\frac{1}{\varepsilon}$ and for $\mu$.

By changing the position of Fourier coefficients in the final results of TE case, we get

$$\omega^2_\pm = (K_z^2 + K_y^2)\left(\frac{\varepsilon_0 \mp |\varepsilon_1|}{\mu_0 \mp |\mu_1|}\right)$$  \hspace{1cm} (2.76)

These are the first order spectral band edge frequencies. Now, for the $\ell$th order, this can be written as:

$$\omega^2_\pm = (K_z^2 + K_y^2)\left(\frac{\varepsilon_0 \mp |\varepsilon_\ell|}{\mu_0 \mp |\mu_\ell|}\right)$$  \hspace{1cm} (2.77)

where $\ell = \pm 1, \pm 2, \pm 3, \ldots$ and the central frequency ($\omega_0$) as:

$$\omega_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}(K_z^2 + K_y^2)} \left(1 - \frac{|\varepsilon_1||\mu_1|}{4\varepsilon_0\mu_0}\right)$$  \hspace{1cm} (2.78)

For the $\ell$th order this can be written as:

$$\omega_0 = \sqrt{\frac{\varepsilon_0}{\mu_0}(K_z^2 + K_y^2)} \left(1 - \frac{|\varepsilon_\ell||\mu_\ell|}{4\varepsilon_0\mu_0}\right)$$  \hspace{1cm} (2.79)

The value of $K_z$ for first order is $\frac{\pi}{\Lambda}$ and for the $\ell$th order it is $\frac{\ell\pi}{\Lambda}$. The band gap for first order and for $\ell$th order can be written as:

$$\Delta \omega_{gap} = \sqrt{\frac{\varepsilon_0}{\mu_0}(K_z^2 + K_y^2)} \left(\frac{|\mu_1|}{\mu_0} - \frac{|\varepsilon_1|}{\varepsilon_0}\right)$$  \hspace{1cm} (2.80)

and

$$\Delta \omega_{gap} = \sqrt{\frac{\varepsilon_0}{\mu_0}(K_z^2 + K_y^2)} \left(\frac{|\mu_\ell|}{\mu_0} - \frac{|\varepsilon_\ell|}{\varepsilon_0}\right)$$  \hspace{1cm} (2.81)
In this chapter, we give the general mathematical calculations by using the Fourier Series method and compute the Bloch wave number for the propagating and evanescent waves in the six layers one-dimensional dielectric-magnetic photonic crystals. Both the axial propagation (3.2) and oblique propagation (3.2) are discussed.
3.1 Fourier Coefficients

Let us consider a photonic crystal with six layers in each unit cells. The thickness of each layer is \(d_1,d_2\) and so on to \(d_6\) and the permittivity constants with the different values of \(\epsilon_1 = 3.5, \epsilon_2 = 4, \epsilon_3 = 3.5, \epsilon_4 = 2, \epsilon_5 = 2\) and \(\epsilon_6 = 2.5\) and also the different values of permeabilities \(\mu_1 = 2.5, \mu_2 = 3, \mu_3 = 2.5, \mu_4 = 1.5, \mu_5 = 1\) and \(\mu_6 = 1.5\). The values of all the thicknesses is 1.

The zeroth and the \(\ell\)th order Fourier coefficients for the permittivity are

\[
\epsilon_0 = \frac{1}{\Lambda} \left[ \sum_i \epsilon_i d_i \right] \quad (3.1)
\]

and

\[
\epsilon_{\ell} = \frac{i}{2\pi \ell} \left[ e^{-i\ell d_1 g} (\epsilon_1 - \epsilon_2) - \epsilon_1 + e^{-i\ell (d_1 + d_2) g} (\epsilon_2 - \epsilon_3) + e^{-i\ell (d_1 + d_2 + d_3) g} (\epsilon_3 - \epsilon_4) \right. \\
+ \left. \frac{i}{2\pi \ell} \left[ e^{-i\ell (d_1 + d_2 + d_3 + d_4) g} (\epsilon_4 - \epsilon_5) + e^{-i\ell (d_1 + d_2 + d_3 + d_4 + d_5) g} (\epsilon_5 - \epsilon_6) + \epsilon_6 \right] \right) \quad (3.2)
\]

where

\[
g = \frac{2\pi}{\Lambda} \quad (3.3)
\]

respectively, the zeroth and \(\ell\)th order Fourier coefficients of inverse of permeability are

\[
\mu_0 = \frac{1}{\Lambda} \left[ \sum_i \mu_i d_i \right] \quad (3.4)
\]

and

\[
\mu_{\ell} = \frac{i}{2\pi \ell} \left[ e^{-i\ell d_1 g} \left( \frac{1}{\mu_1} - \frac{1}{\mu_2} \right) - \frac{1}{\mu_1} + e^{-i\ell (d_1 + d_2) g} \left( \frac{1}{\mu_2} - \frac{1}{\mu_3} \right) + e^{-i\ell (d_1 + d_2 + d_3) g} \left( \frac{1}{\mu_3} - \frac{1}{\mu_4} \right) \right. \\
+ \left. \frac{i}{2\pi \ell} \left[ e^{-i\ell (d_1 + d_2 + d_3 + d_4) g} \left( \frac{1}{\mu_4} - \frac{1}{\mu_5} \right) + e^{-i\ell (d_1 + d_2 + d_3 + d_4 + d_5) g} \left( \frac{1}{\mu_5} - \frac{1}{\mu_6} \right) + \frac{1}{\mu_6} \right] \right] \quad (3.5)
\]

respectively. The zeroth and \(\ell\)th order Fourier coefficients of permeability are

\[
\mu_0 = \frac{1}{\Lambda} \left[ \sum_i \mu_i d_i \right] \quad (3.6)
\]

and

\[
\mu_{\ell} = \frac{i}{2\pi \ell} \left[ e^{-i\ell d_1 g} (\mu_1 - \mu_2) - \mu_1 + e^{-i\ell (d_1 + d_2) g} (\mu_2 - \mu_3) + e^{-i\ell (d_1 + d_2 + d_3) g} (\mu_3 - \mu_4) \right. \\
+ \left. \frac{i}{2\pi \ell} \left[ e^{-i\ell (d_1 + d_2 + d_3 + d_4) g} (\mu_4 - \mu_5) + e^{-i\ell (d_1 + d_2 + d_3 + d_4 + d_5) g} (\mu_5 - \mu_6) + \mu_6 \right] \right] \quad (3.7)
\]
and respectively. The zeroth and the \( \ell \)th order Fourier coefficients of inverse of permittivity are

\[
\bar{\varepsilon}_0 = \frac{1}{\Lambda} \left[ \sum_i \bar{\varepsilon}_i d_i \right] \quad (3.8)
\]

and

\[
\bar{\varepsilon}_\ell = \frac{i}{2\pi \ell} \left[ e^{-i\ell d_1 g} \left( \frac{1}{\varepsilon_1} - \frac{1}{\varepsilon_2} \right) - \frac{1}{\varepsilon_1} + e^{-i\ell(d_1+d_2)g} \left( \frac{1}{\varepsilon_2} - \frac{1}{\varepsilon_3} \right) + e^{-i\ell(d_1+d_2+d_3)g} \left( \frac{1}{\varepsilon_3} - \frac{1}{\varepsilon_4} \right) \right] + \frac{i}{2\pi \ell} \left[ e^{-i\ell(d_1+d_2+d_3+d_4)g} \left( \frac{1}{\varepsilon_4} - \frac{1}{\varepsilon_5} \right) + e^{-i\ell(d_1+d_2+d_3+d_4+d_5)g} \left( \frac{1}{\varepsilon_5} - \frac{1}{\varepsilon_6} \right) + \frac{1}{\varepsilon_6} \right] \quad (3.9)
\]

### 3.2 Results

Using the Fourier coefficients in Eqs. (2.64) and (2.76) we computed the band edge frequencies which are in Figs. (3.3) and (3.4) and showing that as we are going towards higher order band, the gap between two bands is going to be small. The central frequencies are approximately equal to the rigorous results from [14]. Our approximate results have good agreement with the rigorous results when \( k_y \) is small.

![Figure 3.3: Band gap for TE propagation](image1)

![Figure 3.4: Band gap for TM propagation](image2)

### 3.3 Concluding Remarks

We presented the general results for the computation of Bloch wave number for the propagating and evanescent wave in the six layers one-dimensional dielectric-magnetic photonic crystals having periodically varying permittivity and permeability. Axial and oblique propagation were considered for both the TE and TM polarization.
The band gaps and central frequencies were computed by using the coupled-mode theory. Finally, the numerical results were also computed which have good agreement with exact results when propagation is nearly axial.


