Linear Transformations and Signal Estimation in the Joint Spatial-Slepian Domain

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***Abstract*—We develop a framework for generalized linear transformations of the joint spatial-Slepian domain representation of signals on the sphere. Such a representation is enabled by the spatial-Slepian transform on the sphere. We formulate a least-square signal estimation framework for reconstruction of the spherical signal from the modified (transformed) spatial-Slepian representation specified by the spatial-Slepian transformation kernel. We specialize the form of the kernel to present analytical expressions for the multiplicative and convolutive transformations, and use the latter to present illustrations on a Mars topography map.**

***Index Terms*—2-sphere, Slepian functions, spatial-Slepian transform, $\mathbb{SO}(3)$ rotation group, spherical harmonics.**

I. Introduction

SIGNAL processing methods can be viewed from different perspectives, but their chief objective has always been to manipulate and transform signals to extract useful information out of them. Developing a signal processing framework depends on the mathematical representation and hence, on the nature of the underlying signals. In this work, we consider signals that are defined on the spherical manifold. Referred to as spherical signals, these find applications in different fields of science and engineering, such as cosmology [1]–[3], planetary sciences [4], [5], geophysics [6], [7], acoustics [8], [9], medical imaging [10], antenna theory [11] and wireless communication [12]. A popular choice of basis functions on the spherical manifold has been the spherical harmonic functions, which result in the spectral domain representation of spherical signals through the spherical harmonic transform.

Another choice of basis functions for spherical signals are the Slepian functions, which are obtained by solving the spatial-spectral concentration problem on a local region on the sphere [13], [14]. Unlike spherical harmonics, which are defined globally, characteristics of the Slepian functions depend on the shape of the underlying region on the sphere, due to which a subset of the Slepian functions (which are well-optimally concentrated within the local spherical region) can be used to probe local contents of the signal. Motivated by this idea, a joint spatial-Slepian domain representation through the spatial-Slepian transform has been proposed in [15]. Such a representation maps the signal content onto the $\mathbb{SO}(3)$ rotation group across different Slepian scales.

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In this work, the problem of manipulating signals in the joint spatial-Slepian domain is addressed. In particular, we restrict our framework to linear transformations of the joint spatial-Slepian domain representation of spherical signals. Although the spatial-Slepian transform is invertible, there may not exist a spherical signal corresponding to modified spatial-Slepian domain representation. Hence, in addition to formulating a generalized framework for linear transformations in the joint spatial-Slepian domain through spatial-Slepian transformation kernel, we present a least-square estimate for the modified signal on the sphere in Section III. Furthermore, we specialize the form of the spatial-Slepian transformation kernel to present analytical expressions for the multiplicative and convolutive transformations. In Section IV, we use the 1D Gaussian function, along with a more sophisticated optimal filter function, to model the spectral representation of the convolutive transformation kernel and present illustrations on a bandlimited Mars topography map. However, before presenting these formulations, we review the required mathematical background for analyzing signals defined on the sphere and $\mathbb{SO}(3)$ rotation group in the next section.

II. Mathematical Preliminaries

A. Signal Analysis on the Sphere

Two dimensional spherical manifold, called 2-sphere, unit sphere or just sphere, is a set denoted by $\mathbb{S}^2$ and defined as $\mathbb{S}^2 \triangleq \{\widehat{\boldsymbol{x}} \in \mathbb{R}^3 : |\widehat{\boldsymbol{x}}| = 1\}$, where $\widehat{\boldsymbol{x}} \equiv \widehat{\boldsymbol{x}}(\theta,\phi) \triangleq (\sin\theta\cos\phi, \sin\theta\sin\phi, \cos\theta)^{\mathrm{T}}$ is a unit vector representing a point on $\mathbb{S}^2$, $|\cdot|$ denotes the Euclidean norm, and $(\cdot)^{\mathrm{T}}$ represents vector transpose. Here $\theta \in [0,\pi]$ is called colatitude, which is measured from positive z-axis, whereas $\phi \in [0, 2\pi)$ is called longitude, which is measured from positive x-axis in the $x - y$ plane. Square-integrable and complex-valued functions defined on the sphere constitute a Hilbert space, denoted by $L^2(\mathbb{S}^2)$, which is equipped with the following inner product

$$\langle f, g\rangle_{\mathbb{S}^2} \triangleq \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}})\overline{g(\widehat{\boldsymbol{x}})}ds(\widehat{\boldsymbol{x}}), \quad \int_{\mathbb{S}^2} \equiv \int_{\phi=0}^{2\pi}\int_{\theta=0}^{\pi}, \tag{1}$$

where $f, g \in L^2(\mathbb{S}^2)$, $\overline{(\cdot)}$ denotes complex conjugate, and $ds(\widehat{\boldsymbol{x}}) = \sin\theta d\theta d\phi$. Inner product in (1) induces norm of the function $f(\widehat{\boldsymbol{x}})$ as $\|f\|_{\mathbb{S}^2} \triangleq \sqrt{\langle f, f\rangle_{\mathbb{S}^2}}$ and its energy is defined as $\langle f, f\rangle_{\mathbb{S}^2}$. Finite energy functions are referred to as signals.

The Hilbert space $L^2(\mathbb{S}^2)$ has a complete set of orthonormal basis functions, called spherical harmonics, denoted by $Y_\ell^m(\widehat{\boldsymbol{x}})$ for integer degree $\ell \geq 0$ and integer order $|m| \leq \ell$ [16]. Therefore, we can express any signal $f \in L^2(\mathbb{S}^2)$ as

$$f(\widehat{\boldsymbol{x}}) = \sum_{\ell,m}^{\infty} (f)_\ell^m Y_\ell^m(\widehat{\boldsymbol{x}}), \quad \sum_{\ell,m}^{\infty} \equiv \sum_{\ell=0}^{\infty}\sum_{m=-\ell}^{\ell}, \tag{2}$$

where $(f)_\ell^m \triangleq \langle f, Y_\ell^m \rangle_{\mathbb{S}^2}$ is the spherical harmonic (spectral) coefficient of degree ℓ and order m, which gives the spectral representation of the signal $f(\widehat{\boldsymbol{x}})$. We consider $f(\widehat{\boldsymbol{x}})$ to be bandlimited to degree L if $(f)_\ell^m = 0, \forall\, \ell \geq L, |m| \leq \ell$.

B. Signal Analysis on the $\mathbb{SO}(3)$ Rotation Group

Special orthogonal rotation group, denoted by $\mathbb{SO}(3)$, is defined as the set $\mathbb{SO}(3) \triangleq \{\mathbf{R}^{zyz}(\rho) : \det(\mathbf{R}^{zyz}(\rho)) = 1\}$, where $\det(\cdot)$ evaluates determinant of the matrix and

$$\mathbf{R}^{zyz}(\rho) \triangleq \mathbf{R}_z(\varphi)\,\mathbf{R}_y(\vartheta)\,\mathbf{R}_z(\omega), \ \ \rho \triangleq (\varphi, \vartheta, \omega). \tag{3}$$

Here $\varphi \in [0, 2\pi)$, $\vartheta \in [0, \pi]$, $\omega \in [0, 2\pi)$ are the Euler angles which represent rotations on the sphere around z, y and z axes respectively, in the right-handed convention. The set of complex-valued and square-integrable functions defined on the $\mathbb{SO}(3)$ rotation group forms a Hilbert space $L^2(\mathbb{SO}(3))$, which is equipped with the following inner product

$$\langle f, g \rangle_{\mathbb{SO}(3)} \triangleq \int_{\mathbb{SO}(3)} f(\rho)\overline{g(\rho)} d\rho, \int_{\mathbb{SO}(3)} \equiv \int_{\varphi=0}^{2\pi} \int_{\vartheta=0}^{\pi} \int_{\omega=0}^{2\pi}, \tag{4}$$

where $f, g \in L^2(\mathbb{SO}(3))$ and $d\rho \triangleq d\varphi \sin\vartheta d\vartheta d\omega$ is the invariant measure on $\mathbb{SO}(3)$. The inner product in (4) induces norm of the function $f(\rho)$ as $\|f\|_{\mathbb{SO}(3)} \triangleq \sqrt{\langle f, f \rangle_{\mathbb{SO}(3)}}$ and energy of $f(\rho)$ is given by $\langle f, f \rangle_{\mathbb{SO}(3)}$.

The Hilbert space $L^2(\mathbb{SO}(3))$ has a complete set of orthogonal basis functions, called Wigner-D functions, denoted by $D^\ell_{m,m'}(\rho)$ for integer degree $\ell \geq 0$ and integer orders $|m|, |m'| \leq \ell$, which satisfy [16]

$$\langle D^\ell_{m,m'}, D^p_{q,q'} \rangle_{\mathbb{SO}(3)} = c_\ell \delta_{\ell,p}\delta_{m,q}\delta_{m,'q'}, \ c_\ell \triangleq \frac{8\pi^2}{2\ell+1}. \tag{5}$$

As a result, we can represent any signal $f \in L^2(\mathbb{SO}(3))$ as

$$f(\rho) = \sum_{\ell,m,m'}^{\infty} (f)^\ell_{m,m'} \overline{D^\ell_{m,m'}(\rho)}, \ \sum_{\ell,m,m'}^{\infty} \equiv \sum_{\ell,m}^{\infty} \sum_{m'=-\ell}^{\ell}, \tag{6}$$

where $f^\ell_{m,m'} \triangleq c_\ell^{-1} \langle f, \overline{D^\ell_{m,m'}} \rangle_{\mathbb{SO}(3)}$ is the $\mathbb{SO}(3)$ harmonic (spectral) coefficient of degree ℓ and orders m, m', and forms the spectral representation of the signal $f(\rho)$. We consider $f(\rho)$ to be bandlimited to degree L if $(f)^\ell_{m,m'} = 0, \forall\, \ell \geq L, |m|, |m'| \leq \ell$.

III. Linear Transformations and Signal Estimation in the Joint Spatial-Slepian Domain

We consider a bandlimited signal $f \in L^2(\mathbb{S}^2)$ and obtain its joint spatial-Slepian domain representation $F_{g_\alpha} \in L^2(\mathbb{SO}(3))$ through the spatial-Slepian transform on the sphere. The problem under consideration is to formulate a framework for linear transformations of the spatial-Slepian representation $F_{g_\alpha}(\rho)$ and to obtain the transformed signal on the sphere.

A. Spatial-Slepian Transform

Considering a region $R \subset \mathbb{S}^2$, joint spatial-Slepian domain representation of a signal $f \in L^2(\mathbb{S}^2)$, having bandlimit L_f, is given by the spatial-Slepian transform, defined as [15]

$$F_{g_\alpha}(\rho) \triangleq \langle f, \mathcal{D}_\rho g_\alpha \rangle_{\mathbb{S}^2} = \int_{\mathbb{S}^2} f(\widehat{\boldsymbol{x}}) \overline{(\mathcal{D}_\rho g_\alpha)(\widehat{\boldsymbol{x}})} ds(\widehat{\boldsymbol{x}}) \tag{7}$$

for $\alpha \in [1, N_R]$, where $F_{g_\alpha}(\rho)$ is called the spatial-Slepian coefficient at Slepian scale α, $\mathcal{D}_\rho \equiv \mathcal{D}(\varphi, \vartheta, \omega)$ is the rotation operator, corresponding to the rotation characterized by rotation matrix in (3), whose action is given by [16]

$$(\mathcal{D}_\rho g_\alpha)(\widehat{\boldsymbol{x}}) = \sum_{p,q}^{L_g-1} \left(\sum_{q'=-p}^{p} D^p_{q,q'} (g_\alpha)^{q'}_p \right) Y^q_p(\widehat{\boldsymbol{x}}), \tag{8}$$

$g_\alpha(\widehat{\boldsymbol{x}})$ is a Slepian function, assumed bandlimited to degree L_g, and N_R is the spherical Shannon number given by [14]

$$N_R = \frac{L_g^2}{4\pi} \int_R ds(\widehat{\boldsymbol{x}}). \tag{9}$$

Using expansion of signals in (2), along with (8) and orthonormality of spherical harmonics on the sphere, spatial-Slepian coefficients in (7) can be rewritten in the following form

$$F_{g_\alpha}(\rho) = \sum_{\ell,m}^{L_f-1} \psi_{\alpha,\ell m}(\rho)(f)^m_\ell, \tag{10}$$

where[1]

$$\psi_{\alpha,\ell m}(\rho) \triangleq \sum_{m'=\ell}^{\ell} \overline{(g_\alpha)^{m'}_\ell}\, \overline{D^\ell_{m,m'}(\rho)}, \tag{11}$$

from which we observe that

$$(F_{g_\alpha})^\ell_{m,m'} = \frac{1}{c_\ell} \langle F_{g_\alpha}, \overline{D^\ell_{m,m'}} \rangle_{\mathbb{SO}(3)} = (f)^m_\ell \overline{(g_\alpha)^{m'}_\ell}, \tag{12}$$

which can be inverted to obtain the spherical harmonic coefficients of the signal f.

B. Linear Transformations in Joint Spatial-Slepian Domain

We define linear transformation of the spatial-Slepian coefficient of a signal $f \in L^2(\mathbb{S}^2)$, to generalize the framework of spatial-Slepian transform, as

$$\nu_{g_\alpha}(\rho) \triangleq \sum_{\beta=1}^{N_R} \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho, \rho_1) F_{g_\beta}(\rho_1)\, d\rho_1, \tag{13}$$

where $\zeta_{\alpha,\beta}(\rho, \rho_1)$ is the spatial-Slepian transformation kernel and $\nu_{g_\alpha} \in L^2(\mathbb{SO}(3))$ is the modified spatial-Slepian representation of the signal $f(\widehat{\boldsymbol{x}})$. For $\nu_{g_\alpha}(\rho)$ to be an admissible spatial-Slepian representation, there must exist a signal $v \in L^2(\mathbb{S}^2)$, bandlimited to degree L_v, such that

$$(v)^q_p = \frac{(\nu_{g_\beta})^p_{q,q'}}{\overline{(g_\beta)^{q'}_p}} = \frac{1}{c_p} \frac{1}{\overline{(g_\beta)^{q'}_p}} \int_{\mathbb{SO}(3)} \nu_{g_\beta}(\rho) D^p_{q,q'}(\rho) d\rho. \tag{14}$$

Combining (10) with (14), we get the following condition for $\nu_{g_\alpha}(\rho)$ to be an admissible spatial-Slepian representation

$$\overline{(g_\beta)^{q'}_p} \nu_{g_\alpha}(\rho) = \sum_{p,q}^{L_v-1} \frac{1}{c_p} \psi_{\alpha,pq}(\rho) \langle \nu_{g_\beta}, \overline{D^p_{q,q'}} \rangle_{\mathbb{SO}(3)}. \tag{15}$$

C. Least-Square Signal Estimation

The modified spatial-Slepian representation $\nu_{g_\alpha}(\rho)$ can be inverted, using (12), to obtain the spectral coefficients of the corresponding modified spherical signal $v \in L^2(\mathbb{S}^2)$ (assumed bandlimited to degree L_v), only if $\nu_{g_\alpha}(\rho)$ is an admissible representation. For $\nu_{g_\alpha}(\rho)$ being an inadmissible representation, we

[1] We assume that $\min\{L_f, L_g\} = L_f$. For details, please refer to [15].

present a least-square spectral estimate of the modified spherical signal v, by minimizing the following squared error

$$\mathcal{E}_{\rm s} = \sum_{\alpha=1}^{N_R} \|\nu_{g_\alpha}(\rho) - \sum_{p,q}^{L_v-1} \psi_{\alpha,pq}(\rho)(v)_p^q\|^2_{\mathbb{SO}(3)} \qquad (16)$$

with respect to $(v)_p^q$, in the theorem given below.

Theorem 1: Let a spatial-Slepian kernel $\zeta_{\alpha,\beta}(\rho,\rho_1)$ transform the spatial-Slepian coefficient of a signal $f \in L^2(\mathbb{S}^2)$, bandlimited to degree L_f, according to the linear transformation defined in (13), to give $\nu_{g_\alpha}(\rho)$ as an inadmissible modified spatial-Slepian representation. Defining

$$E_{p,\alpha} \triangleq \sum_{q'=-p}^{p} \left|(g_\alpha)_p^{q'}\right|^2 \qquad (17)$$

as the energy per degree of $g_\alpha(\widehat{\boldsymbol{x}})$, spectral estimate of the modified spherical signal, which minimizes the joint spatial-Slepian domain squared error, given in (16), is obtained as

$$(v)_p^q = \sum_{\ell,m}^{L_f-1} \Upsilon_{pq,\ell m}(f)_\ell^m, \quad 0 \le p \le L_v - 1, |q| \le p, \quad (18)$$

where L_v is the bandlimit of the estimate $v(\widehat{\boldsymbol{x}})$ and

$$\Upsilon_{pq,\ell m} = \frac{1}{c_p \sum\limits_{\alpha=1}^{N_R} E_{p,\alpha}} \sum_{\alpha,\beta=1}^{N_R} \sum_{q'=-p}^{p} (g_\alpha)_p^{q'} \sum_{m'=-\ell}^{\ell} \overline{(g_\beta)_\ell^{m'}} \times \int_{\mathbb{SO}(3)} \int_{\mathbb{SO}(3)} \zeta_{\alpha,\beta}(\rho,\rho_1) D^p_{q,q'}(\rho) \overline{D^\ell_{m,m'}(\rho_1)} d\rho_1 d\rho. \qquad (19)$$

Proof: Expanding the squared error in (16) using the definition of norm of signals defined on the $\mathbb{SO}(3)$ rotation group, differentiating the result with respect to $(v)_p^q$ and setting it to zero, we get

$$\sum_{\alpha=1}^{N_R} \int_{\mathbb{SO}(3)} \overline{\psi_{\alpha,pq}(\rho)} \left(\sum_{p',q'}^{L_v-1} \psi_{\alpha,p'q'}(\rho)(v)_{p'}^{q'} - \nu_{g_\alpha}(\rho) \right) d\rho = 0. \qquad (20)$$

Using (11) and (5), we can solve the first integral in (20) as

$$\int_{\mathbb{SO}(3)} \overline{\psi_{\alpha,pq}(\rho)} \psi_{\alpha,p'q'}(\rho) d\rho = c_p \sum_{q''=-p}^{p} \left|(g_\alpha)_p^{q''}\right|^2 \delta_{p,p'}\delta_{q,q'},$$

which gives the following expression for $(v)_p^q$

$$(v)_p^q = \frac{1}{c_p \sum\limits_{\alpha=1}^{N_R} E_{p,\alpha}} \sum_{\alpha=1}^{N_R} \int_{\mathbb{SO}(3)} \overline{\psi_{\alpha,pq}(\rho)} \nu_{g_\alpha}(\rho) d\rho, \qquad (21)$$

where $E_{p,\alpha}$ is defined in (17). The integral in (21) can be solved by employing the expressions for $\psi_{\alpha,pq}$ in (11) and ν_{g_α} in (13) to obtain the linear system given in (18). ∎

We note that the spectral estimate in (18) is a generalization of the formulation of inverse spatial-Slepian transform in [15].

D. Example Linear Transformations

1) Admissible Transformation: The simplest linear transformation, resulting in an admissible spatial-Slepian representation, is obtained by choosing the transformation kernel as

$$\zeta_{\alpha,\beta}(\rho,\rho_1) = C\,\delta_{\alpha,\beta}\,\delta(\rho-\rho_1), \qquad (22)$$

where C is some complex number, $\delta_{\alpha,\beta}$ is the Kronecker delta function and $\delta(\rho-\rho_1)$ is the Dirac delta function, defined as

$$\delta(\rho-\rho_1) \triangleq (\sin\vartheta)^{-1}\delta(\varphi-\varphi_1)\delta(\vartheta-\vartheta_1)\delta(\omega-\omega_1). \quad (23)$$

It is trivial to show that the modified spatial-Slepian representation corresponding to the transformation kernel in (22) satisfies the admissibility condition in (15). The spectral estimate in this case simply becomes

$$(v)_p^q = \sum_{\ell,m}^{L_f-1} \Upsilon_{pq,\ell m}(f)_\ell^m = \sum_{\ell,m}^{L_f-1} \delta_{\ell,p}\delta_{m,q}(f)_\ell^m = (f)_p^q. \quad (24)$$

2) Inadmissible Transformations: Let the transformation kernel be defined as

$$\zeta^{\rm M}_{\alpha,\beta}(\rho,\rho_1) \triangleq \zeta^{\rm M}_\alpha(\rho)\delta_{\alpha,\beta}\,\delta(\rho-\rho_1), \qquad (25)$$

then the modified representation, given by

$$\nu_{g_\alpha}(\rho) = \zeta^{\rm M}_\alpha(\rho) F_{g_\alpha}(\rho), \qquad (26)$$

is called the multiplicative transformation. Alternatively, defining the transformation kernel as

$$\zeta^{\circledast}_{\alpha,\beta}(\rho,\rho_1) \triangleq \zeta^{\circledast}_\alpha(\rho\rho_1^{-1})\delta_{\alpha,\beta} \qquad (27)$$

results in the following convolutive transformation

$$\nu_{g_\alpha}(\rho) = \int_{\mathbb{SO}(3)} \zeta^{\circledast}_\alpha(\rho\rho_1^{-1}) F_{g_\alpha}(\rho_1) d\rho_1 = \left(\zeta^{\circledast}_\alpha \circledast F_{g_\alpha}\right)(\rho), \qquad (28)$$

where $\circledast$ denotes convolution of signals defined on the $\mathbb{SO}(3)$ rotation group [17], [18]. It can be observed that modified representations in (26) and (28) do not satisfy (15) in general and hence, may not be admissible spatial-Slepian representations.

The convolutive transformation in (28) represents filtering of the spatial-Slepian coefficient $F_{g_\alpha}(\rho)$ with the joint spatial-Slepian domain filter function $\zeta^{\circledast}_\alpha(\rho)$, which can be assumed bandlimited to degree L_f. Using the spectral representation of convolution of $\mathbb{SO}(3)$ signals [18], along with the orthogonality constraint in (5), the estimated signal is given by

$$(v)_p^q = c_p \left(\sum_{\alpha=1}^{N_R} E_{p,\alpha} \right)^{-1} \sum_{q'=-p}^{p} \sum_{\alpha=1}^{N_R} E_{p,\alpha} (\zeta^{\circledast}_\alpha)^p_{q,q'} (f)_p^{q'}. \quad (29)$$

IV. Analysis

As a simple application of the framework formulated in Section III, we consider $f(\widehat{\boldsymbol{x}})$ to be a noise-contaminated observation of a source signal $s \in L^2(\mathbb{S}^2)$, i.e., $f(\widehat{\boldsymbol{x}}) = s(\widehat{\boldsymbol{x}}) + z(\widehat{\boldsymbol{x}})$, where $z \in L^2(\mathbb{S}^2)$ is a realization of a zero-mean and anisotropic noise process on the sphere. The source and noise signals are assumed to be uncorrelated, i.e., $\mathbb{E}\{(s)_\ell^m \overline{(z)_{\ell'}^{m'}}\} = 0, \forall\, \ell, \ell', |m| \le \ell, |m'| \le \ell'$, where $\mathbb{E}\{\cdot\}$ represents the expectation operator. Then, we use the convolutive transformation kernel in (27) to obtain $(v)_p^q$ in (29) as the spectral estimate of the source signal $s(\widehat{\boldsymbol{x}})$. We gauge the performance of the filter using the signal to noise ratio (SNR), defined for a signal $d \in L^2(\mathbb{S}^2)$ as

$$\text{SNR}^d \triangleq 10 \log \frac{\|s(\widehat{\boldsymbol{x}})\|^2_{\mathbb{S}^2}}{\|d(\widehat{\boldsymbol{x}}) - s(\widehat{\boldsymbol{x}})\|^2_{\mathbb{S}^2}}. \qquad (30)$$

Hence, SNR^f and SNR^v represent input and output SNRs respectively. We compute the signal estimate by smoothing the noise-contaminated observation in the spectral domain through a Gaussian kernel at each Slepian scale, i.e., we define the convolutive transformation kernel as

$$(\zeta_\alpha^{\circledast})^p_{q,q'} \triangleq c_p^{-1} e^{\frac{-[p(p+1)+q]^2\alpha^2}{L_f^4}} \delta_{q,q'}, \tag{31}$$

to obtain the spectral estimate, from (29), as a weighted sum of smoothed noise-contaminated observations, i.e.,

$$(v_{\mathrm{G}})^q_p = \left(\sum_{\alpha=1}^{N_R} E_{p,\alpha}\right)^{-1} \sum_{\alpha=1}^{N_R} E_{p,\alpha}\, e^{\frac{-[p(p+1)+q]^2\alpha^2}{L_f^4}} (f)^q_p. \tag{32}$$

Alternatively, we can define a more sophisticated convolutive transformation kernel by minimizing the following mean-square error with respect to $\overline{(\zeta_\alpha^{\circledast})^p_{q,q'}}$

$$\mathcal{E}_{\mathrm{ms}} = \sum_{p,q}^{L_f-1} \mathbb{E}\left\{\left|(v)^q_p - (s)^q_p\right|^2\right\}. \tag{33}$$

Using (29) to expand $\mathcal{E}_{\mathrm{ms}}$ in (33), simplifying through the fact that noise is uncorrelated with the source signal, differentiating with respect to the conjugate of spectral representation of the convolutive kernel, and putting the result equal to zero, we obtain an under-determined system of equations for the spectral representation of convolutive transformation kernel as solution to the following linear system

$$\mathbf{A}(p)\,\mathbf{x}(p,q) = \mathbf{b}(p,q),\ \ 0 \le |q| \le p \le L_f - 1, \tag{34}$$

where the elements of matrix $\mathbf{A}$ and column vectors $\mathbf{b}$, $\mathbf{x}$ are given by

$$A_{k,q'} = c_p(C^s_{pq',pk} + C^z_{pq',pk}),\ \ b_k = C^s_{pq,pk},\ \ |k| \le p,$$

$$x_{q'} = \left(\sum_{\alpha=1}^{N_R} E_{p,\alpha}\right)^{-1} \sum_{\alpha=1}^{N_R} E_{p,\alpha}(\zeta_\alpha^{\circledast})^p_{q,q'},\ \ |q'| \le p. \tag{35}$$

Here $C^d_{pq',pk} = \mathbb{E}\{(d)^{q'}_p \overline{(d)^k_p}\}$ represents the spectral covariance matrix of a signal $d \in L^2(\mathbb{S}^2)$. The expression for $x_{q'}$ in (35) results in infinitely many solutions for $(\zeta_\alpha^{\circledast})^p_{q,q'}$. We choose $(\zeta_\alpha^{\circledast})^p_{q,q'}$ to be independent of the Slepian scale as

$$(\zeta_\alpha^{\circledast})^p_{q,q'} = x_{q'}(p,q), \quad \forall\, \alpha \in [1, N_R], \tag{36}$$

i.e., as the solution of (34) and call it the optimal filter. The *optimal* spectral estimate is then obtained from (29) as

$$(v_{\mathrm{O}})^q_p = c_p \sum_{q'=-p}^{p} x_{q'}(p,q)(f)^{q'}_p, \tag{37}$$

where $x_{q'}$ are found by inverting (34).

Remark 1: We observe a close similarity in the mathematical formulation of the spatial-Slepian transform and the scale-discretized wavelet transform (SDWT) presented in [19], [20], due to which a similar framework of linear transformations can be formulated for SDWT as well. We note that the optimal filter in (34) has been formulated using SDWT in [21].

A. Illustrations

We consider a Mars topography map[2] (processed to have zero mean and unit norm), bandlimited to degree $L_f = 64$, as the source signal $s(\widehat{\boldsymbol{x}})$, and contaminate it with different realizations of zero-mean, uncorrelated and anisotropic Gaussian noise process. We compute the spectral estimate in (32), where $N_R = 8$ represents the spherical Shannon number (rounded to the nearest integer) for an axisymmetric north polar cap region of polar cap angle $\Theta_c = 5°$. We generate 10 realizations of the Gaussian noise process at different noise levels and compute $\mathrm{SNR}^{v_{\mathrm{G}}}$. In Fig. 1, we show the output SNR against the input SNR, both averaged over 10 realizations. Also, shown is the SNR curve for the optimal spectral estimate in (37) using the optimal filter in (34)[3]. As expected, optimal filtering performs better than Gaussian spectral smoothing.

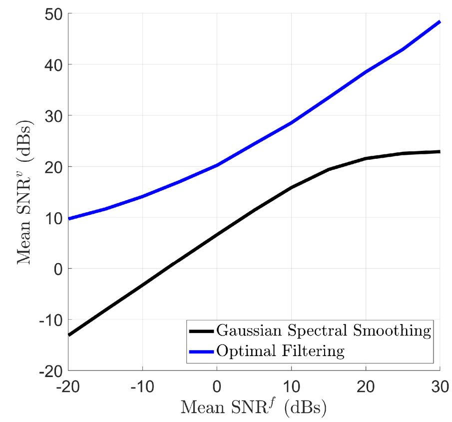

Fig. 1. Output SNR, computed from the spectral estimates in (32) and (37), is averaged over 10 realizations of a zero-mean, uncorrelated and anisotropic Gaussian noise process and shown against average input SNR. Slepian scale is set by the Shannon number for a north polar cap region of angle 5°.

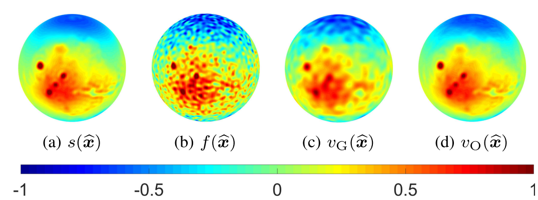

Fig. 2. (a) Mars topography map, (b) noise-contaminated observation having $\mathrm{SNR}^f = 4.9$ dBs, (c) Mars topography map obtained from Gaussian spectral smoothing and (d) Mars topography map reconstructed through optimal filtering.

Fig. 2 shows an illustration for one of the realizations of the zero-mean, uncorrelated and anisotropic Gaussian noise process at $\mathrm{SNR}^f = 4.9$ dBs. Output SNRs for the signal estimates obtained from Gaussian spectral smoothing and optimal filtering are $\mathrm{SNR}^{v_{\mathrm{G}}} = 11.3$ dBs and $\mathrm{SNR}^{v_{\mathrm{O}}} = 24.7$ dBs respectively.

V. Conclusion

We have developed a framework for generalized linear transformations in the joint spatial-Slepian domain for signals on the sphere. Representation of signals in the joint spatial-Slepian domain is obtained from the spatial-Slepian transform on the sphere. We have formulated a least-square signal estimation framework to reconstruct the spherical signal from its modified spatial-Slepian representation specified by the spatial-Slepian transformation kernel. By assuming specific forms for the transformation kernel, we have presented analytical expressions for the multiplicative as well as convolutive transformations. We have demonstrated the utility of the overall framework, using convolutive transformation, on a bandlimited Mars topography map.

[2]http://geoweb.princeton.edu/people/simons/software.html

[3]$C^s_{\ell m,pq} = (s)^m_\ell \overline{(s)^q_p}$ and $C^z_{\ell m,pq} = \sum_{u,t}^{L_f-1} T_{\ell m,ut}\overline{T_{ut,pq}}$, where $T_{\ell m,ut}$ are uniformly distributed in $(-1, 1)$, in both real and imaginary parts.

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