Quantum field theory with a focus on conformal field theory

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Abstract

This work begins by delineating the intuitive meaning of a quantum field, which is then made precise through various mathematical tools. The quantization programs of canonical quantization and path integral formalism are introduced, and important physical quantities called correlators are calculated. The focus of the work is thus kept in such a way that the subset of quantum field theories called conformal field theories (which correlators play an important role in) could conveniently be brought up. The discussion of this subset, in turn, is kept in such a manner that 2-dimensional conformal field theories could easily be introduced.

These 2-dimensional theories are then discussed at great length, and such important topics as primary fields and operator product expansions are broached. Finally, free, massless bosonic and fermionic fields on an infinite cylinder are presented as examples of 2-dimensional conformal field theories, and their various aspects are looked upon. The work ends by studying the famous Wess-Zumino-Witten model as an instance of conformal field theories with Lie algebraic symmetry and showing that the model is indeed a conformal field theory in 2 dimensions via the so-called Sugawara construction.

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What is a quantum field?

A field, in classical field theory, is a quantity defined at every point of spacetime, some good examples being the gravitational and the electromagnetic fields. The primary motivation to introduce such a concept has to do with the notion of *locality*; we want laws of nature that are *local*. Loosely speaking, what it means is that "action at a distance," as seen in the old laws of Newton and Coulomb, is not allowed. It is well known that the theories of Einstein and Maxwell fix these laws by introducing fields to mediate the aforementioned "action" in a local manner. Indeed then, introduction of fields does yield theories that are local, something which constitutes a key motivation to bring fields into quantum mechanics and consider a *quantum* field theory. Before going on to actually formulate and work with such a theory, however, we need to define what we mean by the term *quantum field*, and a good way to do so is to begin by seeing how quantization—promotion of classical quantities, or degrees of freedom, to operators—fits with fields.

As its name implies right away, a quantum field is simply a classical field quantized, and as might easily be guessed, a quantum field theory entails simply the quantization of a classical field. Just as the classical degrees of freedom are promoted to operators in quantum mechanics, so are the field degrees of freedom—or the degrees of freedom of a classical field—promoted to operators in quantum field theory. A degree of freedom in the latter case corresponds to the value of a classical field at a point of spacetime, and since spacetime has infinitely many points, a classical field has infinitely many degrees of freedom. With all of them promoted to operators, we could say that a field in quantum field theory, or a quantum field, is an operator-valued function of points in spacetime. Encapsulating quantization as it applies to classical fields, this simple definition makes clear what a quantum field means and entails. In addition to bringing locality in, however, what motivations do we have to concoct such a contraption? It turns out that we have many, but we look at only some of the most salient ones here. Apart from motivating the need for quantum fields, they further consolidate the definition we gave above.

A major motivation comes from processes that particle numbers change in. Contrary to what one might think, particles could be pretty ephemeral in terms of existence; that is, they could be both created and destroyed. Supported by incontrovertible evidence that comes from large particle colliders (see Fig. 1), this fact makes it absolutely clear that theories lacking a mechanism to deal with changing numbers of particles are bound to be inadequate. What they need is an entity that affords the ability to both create and destroy particles at every point of spacetime, and ordinary quantum mechanics provides us with a major hint as regards this entity, namely that it must accouter every spacetime point with an operator resembling the creation/annihilation ones. Seeking to equip each spacetime point with an operator, however, we note how it is the same as subjecting a classical field to quantization, which promotes the field values to operators everywhere and gives what we referred to as a quantum field above. A quantum field could then also be viewed as an entity that has the capacity to both create and destroy particles at every point of spacetime and adequately describe processes that particle numbers are not conserved in.

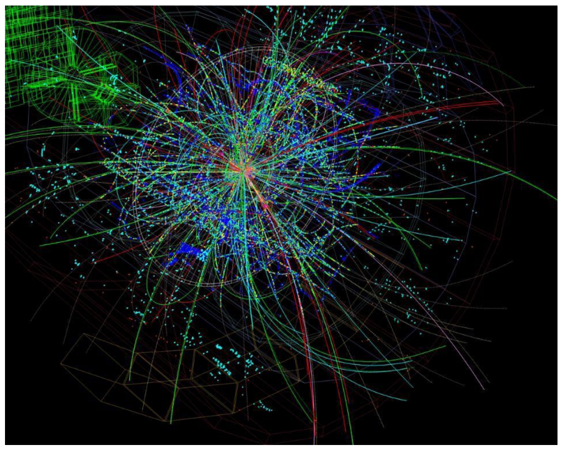


Figure 1: Rich particle production in the Large Hadron Collider (LHC). We need a theory that goes beyond theories with fixed numbers of particles and encapsulates the fact that particles could be both created and destroyed.

Just to give an example of such a process, we note that when a particle of mass m is localized to distances shorter than $\lambda = \hbar/mc$, the uncertainty in its energy is high enough to make particle/antiparticle pairs pop out of vacuum (see Fig. 2). Known as the *Compton wavelength*, the distance $\hbar/mc$ gives the point that the concept of a fixed number of particles completely breaks down at, and it must be remembered that the new particles entering the picture are of great physical consequence; they exert forces, transfer energy, and do almost anything that a particle could. To encapsulate their effects and accurately describe the physics of particles confined to distances smaller than their respective Compton wavelengths, we need a framework embracing the capacity to both create and destroy particles, a framework obtained by introducing such things as quantum fields.

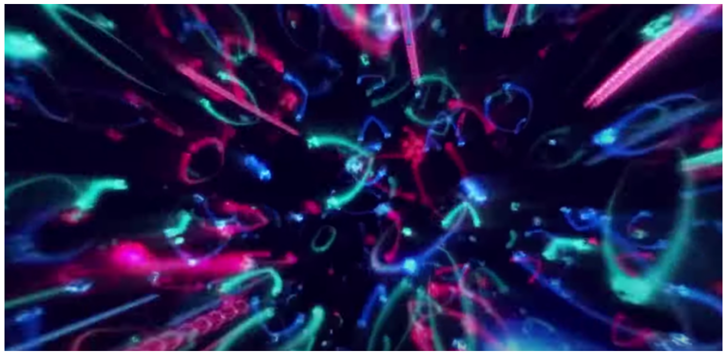


Figure 2: Particle/antiparticle pairs popping into and out of existence in vacuum. Usually taken as being "empty," vacuum teems with particle/antiparticle pairs at small scales, where energy uncertainties are high.

A further spur to consider quantum fields and related paraphernalia comes from what is known as the indistinguishability of elementary particles. Simply put, it refers to the fact that a neutrino produced in the Large Hadron Collider (LHC) is identical—in every respect conceivable—to a neutrino produced in the wake of a supernova billions of light-years away. This observation is just too remarkable to not warrant a solid explanation. What is it after all that ensures such a great deal of precision when it comes to creating elementary particles at different points of spacetime? Even if we posit that the "material" to create elementary particles is the same everywhere, we need some machinery to ensure the aforementioned precision—to ensure that the same amount of "material" is used everywhere and in the right proportions. This universal machinery, spanning all of spacetime, could only come from a quantity defined at every spacetime point, and as we state right at the outset, such a quantity is a field. Having been led to consider some field, we reckon that our field needs to be capable of making particles too, and we thus require that it be an operator field, or, as we have been calling it, a quantum field. It is hence possible to see a quantum field as an invisible swathe too that spans all of spacetime and could make particles of the same kind anywhere and anytime.

We see how naturally the indistinguishability of elementary particles and processes involving particle creation and destruction lead us to consider quantizing a classical field and working with a quantum field. There are other motivations to consider quantum fields as well: the problem of negative energies, bosonic/fermionic statistics, and so on. However, we leave them and their details for more advanced works (and our subsequent chapters) to cover. With what we have done up till now, we could say that we have delineated clearly the meaning of a quantum field, thereby attaining the goal we set for this introductory chapter. In the next chapter, we move to making this meaning precise via mathematics and showing how quantum fields are obtained from the quantization of classical ones.

"Anyone who is not shocked by the quantum theory does not understand it."

Neils Bohr

1

From classical to quantum fields[2,6]

Having seen the intuitive meaning of quantum fields and the motivations we have to study them, we proceed to making things precise in this chapter. Our approach begins with a brief discussion of some useful elements of classical field theory, such as Lagrangian densities, equations of motion, Noether's theorem, and so on. We then introduce the canonical quantization program, discussing it in the context of ordinary quantum mechanics and generalizing it to the realm of classical fields. Finally, we give an explicit demonstration of canonical quantization by applying it to the Klein-

Gordon and Dirac fields. As a freebie toward the end, we discuss what is called the path integral formalism as a quantization scheme alternate and equivalent to the scheme of canonical quantization.

1.1 Classical field theory

As we said in chapter 0, a field in classical field theory is a quantity defined at every point of spacetime. It could, therefore, be written as $\varphi(x)$, where φ represents the field and x represents the spacetime point that the field takes the value $\varphi(x)$ at. It should immediately be clear that we are dealing with infinitely many degrees of freedom here: there is at least one, namely the field value $\varphi(x)$, for every point of spacetime, x.

The fundamental quantity governing dynamics in classical field theory is the *action* S, which is the time integral of something known as the *Lagrangian*. Denoted by L, the Lagrangian itself is taken as being the spatial integral of a *Lagrangian density*, $\mathcal{L}$, which is a functional of the field φ and its derivative $\partial_\mu \varphi$. We could summarize these relations via

$$S = \int dt L(t) = \int dt \int d^3\mathbf{x} \mathcal{L}[\varphi, \partial_\mu \varphi] = \int d^4x \mathcal{L}[\varphi, \partial_\mu \varphi]. \tag{1.1}$$

Since $\mathcal{L}$ usually depends on several fields and their derivatives, it is better to write it as $\mathcal{L}[\varphi_a, \partial_\mu \varphi_a]$ in the integrals above, the Latin subscript a being used to label the fields under consideration. However, all through this work, we drop this Latin subscript and take it as being understood that we could be dealing with several fields at once.

According to the *principle of least action*, the evolution of a system of fields from an initial configuration to some final one occurs along a configuration space "path" that S is stationary for. What it means is that a small variation in that "path" does not change S in any significant way. Mathematically, we could impose this condition in the following manner:

$$\begin{aligned}
0 &= \delta S \\
&= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi + \frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} \delta \left(\partial_\mu \varphi\right) \right) \\
&= \int d^4x \left(\frac{\partial \mathcal{L}}{\partial \varphi} \delta \varphi - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} \right) \delta \varphi + \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} \delta \varphi \right) \right).
\end{aligned} \tag{1.2}$$

The last term here gives us a surface integral over the boundary of the spacetime region of integration. Assuming that the initial and final configurations are given, we could say that $\delta\varphi$ is zero at the temporal beginning and end of this region. If we further assume that $\delta\varphi$ vanishes at the spatial boundary too, then the surface integral yields nothing but zero. It is easy to factor $\delta\varphi$ from the remaining two terms and note that since the integral has to vanish for arbitrary $\delta\varphi$, the quantity multiplying $\delta\varphi$ must vanish at all points in the region. Thus,

$$\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial \left(\partial_\mu \varphi\right)} \right) - \frac{\partial \mathcal{L}}{\partial \varphi} = 0. \tag{1.3}$$

This equation is the *Euler-Lagrange equation of motion* for the field φ, and if $\mathcal{L}$ depends on more than one field, there is one such equation for each.

In going from the action to the Euler-Lagrange equations, we have presented what is commonly known as the Lagrangian formulation of classical field theory, or simply as the *Lagrangian field theory*. It is good in that all expressions it involves are explicitly Lorentz invariant, something meaning that this formulation is particularly suited to relativistic dynamics. However, since our goal is to quantize fields, we want a formulation that is suited to the canonical quantization program, which we introduce later in this chapter. It turns out that such a formulation is what is known as the Hamiltonian formulation of classical field theory, or simply as the *Hamiltonian field theory*. It begins with a field, φ, and its *conjugate momentum field*, π, which is defined as

$$\pi(t, \mathbf{x}) = \frac{\partial \mathcal{L}}{\partial \dot{\varphi}}, \tag{1.4}$$

and then introduces the so-called *Hamiltonian density* $\mathcal{H} = \pi\dot{\varphi} - \mathcal{L}$, which could be integrated over space to obtain the *Hamiltonian* H. Here, we again recall that we could be dealing with more than one field, and in that case, the first term in $\mathcal{H}$ is to be replaced by $\sum_a \pi^a \dot{\varphi}_a$; however, as we did before, we choose to drop the Latin index a and take it as being understood that we could be dealing with several fields. It should be noted that we have sacrificed manifest Lorentz invariance here by singling time out as a special coordinate. In fact, the dynamics of the field φ and its conjugate momentum field π are governed by

$$\dot{\varphi}(t, \mathbf{x}) = \frac{\partial H}{\partial \pi}$$

and

$$\dot{\pi}(t, \mathbf{x}) = -\frac{\partial H}{\partial \varphi},$$

which, unlike the Euler-Lagrange equations, do not look Lorentz invariant at all. Despite this fact, however, the physics they describe has to be the same as that described by the Lorentz invariant Euler-Lagrange equations, for if we start with a relativistic theory, our results ought to be Lorentz invariant even if the intermediate steps are not.

Having presented the Lagrangian and Hamiltonian field theories, we move to the last elements of classical field theory whose discussion we deem indispensable: *symmetries* and *conservation laws*. It is difficult to overstate the importance of symmetries in classical field theory—especially continuous ones as we would see—and the same could be said about conservation laws. Whereas a symmetry could be roughly defined as a transformation that leaves something invariant, a conservation law stipulates that some quantity does not change with time. What is interesting is that the two are intimately linked by a famous theorem known as the Noether theorem, which simply states that to every continuous symmetry corresponds a conserved quantity (or a conservation law). We begin our discussion with a simple derivation of Noether's theorem and then proceed to demonstrating its use by applying it to one of the simplest instances of continuous symmetries in physics, namely spacetime translation.

Considering $\varphi(x)$, we introduce the following continuous transformation.

$$\begin{aligned} x'^{\mu} &= x^{\mu} + \omega_a \frac{\partial x^{\mu}}{\partial \omega_a}; \\ \varphi'(x') &= \varphi(x) + \omega_a \frac{\partial \mathcal{F}}{\partial \omega_a}. \end{aligned} \tag{1.5}$$

The ω_a are infinitesimal parameters controlling the respective changes in x^{μ} and $\mathcal{F}$, where $\mathcal{F}$ is defined in such a way that $\varphi'(x') = \mathcal{F}(\varphi(x))$. It is important to note that the field φ, taken as a mapping from spacetime to some target space $\mathcal{M}$ ($\varphi : \mathbb{R}^4 \to \mathcal{M}$), is affected in two ways: the functional change $\varphi' = \mathcal{F}(\varphi)$ and the argument change $x \to x'$. This way of looking at the transformation in Eq. (1.5) is called the *active* point of view (see Fig. 1.1)—in contrast to the *passive* point of view, which sees the transformation $x \to x'$ merely as a relabeling of coordinates, or the observer's changing perspective.

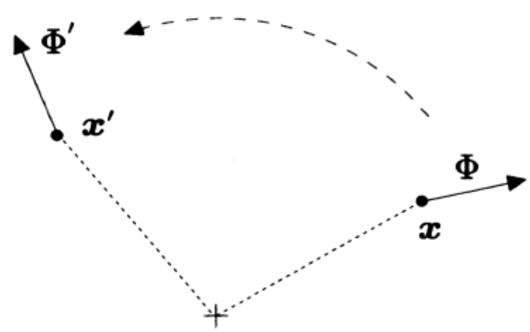


Figure 1.1: An active transformation (here a rotation). The arrows show a vector field that itself undergoes a rotation identical to that of the coordinates. In the passive view, the observer rotates—and in the opposite direction.

It is not surprising that the transformation being considered would transform the action as well, which depends on φ and $\partial_\mu \varphi$ after all. Some transformations of this kind, however, could be such that the transformed action is the same as the untransformed one; that is, the action is left invariant. These transformations comprise a subset of the transformations described by Eq. (1.5) and are known as symmetries (continuous symmetries to be precise). According to Noether's theorem, each of them is associated to a conserved quantity and hence a conservation law. We would use the variation of the action to derive both. Using S' to denote the transformed action, we could write

$$
\begin{aligned}
S' = \int d^4x \left(1 + \partial_\mu \left(\omega_a \frac{\partial x^\mu}{\partial \omega_a}\right)\right) \\
\times \mathcal{L}\left(\varphi + \omega_a \frac{\partial \mathcal{F}}{\partial \omega_a}, \left[\delta^\nu_\mu - \partial_\mu \left(\omega_a \left(\frac{\partial x^\nu}{\partial \omega_a}\right)\right)\right] \left(\partial_\nu \varphi + \partial_\nu \left[\omega_a \left(\frac{\partial \mathcal{F}}{\partial \omega_a}\right)\right]\right)\right).
\end{aligned}
$$

Then, the variation $\delta S = S' - S$ contains terms with no derivatives of the ω_a, and they sum up to zero if the ω_a are rigid (independent of position). With this thing, δS involves only the first derivatives of the ω_a and could be written as

$$
\delta S = -\int d^4x\, j^\mu_a \partial_\mu \omega_a,
$$

where

$$
j^\mu_a = \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\nu \varphi - \delta^\mu_\nu \mathcal{L}\right) \frac{\partial x^\nu}{\partial \omega_a} - \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \frac{\partial \mathcal{F}}{\partial \omega_a}. \tag{1.6}
$$

Known as the *current* associated with the transformation in Eq. (1.5), the quantity j^μ_a is of central importance to Noether's theorem. Integration by parts turns the expression for δS above into

$$
\delta S = \int d^4x\, \partial_\mu j^\mu_a \omega_a.
$$

If the ω_a correspond to a symmetry now, then the variation δS above should equal 0 for any ω_a, something implying that $\partial_\mu j^\mu_a$ should equal 0 for all points in the spacetime integration volume. This very result, expressly,

$$\partial_\mu j^\mu_a = 0, \tag{1.7}$$

is what we refer to as a conservation law, for it requires that a particular quantity stay constant with time, or, in other words, remain conserved. Also known as the *conserved charge*, this quantity turns out to be

$$Q_a = \int d^3\mathbf{x} j^0_a,$$

and it happens to be conserved since its time derivative vanishes:

$$\begin{aligned}\dot{Q}_a &= \int d^3\mathbf{x} \partial_0 j^0_a \\ &= -\int d^3\mathbf{x} \partial_i j^i_a \\ &= -\int j^i_a d\sigma^i,\end{aligned}$$

where $d\sigma^i$ is a surface element at spatial infinity, so given that j^i_a vanishes sufficiently fast as $\mathbf{x} \to \infty$, which is always assumed to be the case, we could say that $\dot{Q}_a = 0$. Now, since $\partial_\mu j^\mu_a = 0$ emerged as a consequence of our assuming that the continuous transformation of Eq. (1.5) comprised a symmetry, we have shown that to every continuous symmetry corresponds a conservation law and a conserved quantity. Put simply, we have derived the Noether theorem and could now go about considering its application to spacetime translation.

If we replace ω_a by ω^γ in $x'^\mu = x^\mu + \omega_a(\partial x^\mu/\partial \omega_a)$, we obtain $x'^\mu = x^\mu + \omega^\mu$, which indicates an infinitesimal spacetime translation by the vector ω^μ; moreover, supposing that the field itself does not undergo any change—other than that imparted to it by the active translation of the spacetime coordinates—we have that $\mathcal{F}(\varphi) = \varphi$, and hence, $\partial\mathcal{F}/\partial\omega_a = 0$. Then, using this fact and that

$\partial x^\mu / \partial \omega_a = \partial x^\mu / \partial \omega^\gamma = \delta^\mu_\gamma$ together with Eq. (1.6), we find the current to be

$$j^\mu_\gamma = \frac{\partial \mathcal{L}}{\partial(\partial_\mu \varphi)} \partial_\gamma \varphi - \delta^\mu_\gamma \mathcal{L}. \tag{1.8}$$

Calling it T^μ_γ instead (for reasons of convention), we see that it satisfies $\partial_\mu T^\mu_\gamma = 0$ with the conserved charge

$$P_\gamma = \int d^3\mathbf{x} T^0_\gamma,$$

which happens to be the four-momentum. That is, translational symmetry in spacetime spells conservation of both energy and momentum, and the conserved charge T^μ_γ is known as the *energy-momentum tensor*, a quantity which holds central importance as subsequent portions of this work would show.

The discussion of Noether's theorem done, we end this section with a brief remark about j^μ_a, namely that its definition is ambiguous to some extent. The expression in Eq. (1.6) is termed "canonical," for there are other admissible expressions as well; in fact, we may add to it the divergence of an antisymmetric tensor without affecting its conservation: $j^\mu_a \rightarrow j^\mu_a + \partial_\nu B^{\nu\mu}_a$, where $B^{\nu\mu}_a = -B^{\mu\nu}_a$. Indeed, $\partial_\mu \partial_\nu B^{\nu\mu}_a = 0$ by antisymmetry, so $\partial_\mu j^\mu_a$ still equals 0.

1.2 Canonical quantization

This section takes the first step toward the central goal of this chapter—going from classical fields to the quantum ones—by presenting the scheme, or program, of *canonical quantization*. It essentially is a recipe that takes one from the dynamics of classical degrees of freedom to the realm of quantum theory. In classical mechanics, for example, it asks us to take the generalized coordinates q_a and their conjugate momenta p^a ($= \partial L / \partial \dot{q}_a$, where L is the Lagrangian of classical mechanics) and promote them to the operators $\hat{q}_a$ and $\hat{p}^a$. Then, it prescribes that we replace the Poisson bracket structure of classical mechanics with the following commutation relations.

$$
\begin{aligned}
[\hat{q}_a, \hat{q}_b] &= \left[\hat{p}^a, \hat{p}^b\right] = 0; \\
\left[\hat{q}_a, \hat{p}^b\right] &= \imath \delta^b_{\ a}.
\end{aligned}
$$

Finally, we are required to put these operators in place of their classical counterparts in the Hamiltonian of classical mechanics and work out the spectrum of the (Hamiltonian) operator that results.

For fields, we do the same, but in lieu of q_a and p^a, we use the fields themselves and their conjugate fields, and after promoting them to operators (operator-valued functions of spacetime to be precise), we define for them what are known as *equal-time commutation relations*. That is,

$$
\begin{aligned}
[\hat{\varphi}(t, \mathbf{x}), \hat{\varphi}(t, \mathbf{y})] &= [\hat{\pi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] = 0; \\
[\hat{\varphi}(t, \mathbf{x}), \hat{\pi}(t, \mathbf{y})] &= \imath \delta^{(3)}(\mathbf{x} - \mathbf{y}).
\end{aligned} \tag{1.9}
$$

It must be noted here that for more than one kind of field, we would also have to define "interfield" commutation relations, and as is conventionally the case, we would set them equal to 0; that is, a nontrivial commutation relation (the second line in Eq. (1.9)) is obtained only from a field and its corresponding conjugate field. With these relations having been defined, we could proceed to evaluate the Hamiltonian for our system,

$$
\hat{H} = \int d^4x \left(\hat{\pi}(x)\dot{\hat{\varphi}}(x) - \hat{\mathcal{L}}\right),
$$

and then go about trying to work out what we called its spectrum above.

As a simple implementation of this scheme of quantization, we present its application to one of the simplest of the systems in all of classical mechanics: an infinitesimal particle of mass m that moves in potential $V(x)$. Calling the coordinate of the particle x and the conjugate momentum p, we consider the operators $\hat{x}$ and $\hat{p}$ and impose the commutation relation $[\hat{x}, \hat{p}] = \imath$. Then, knowing that the Lagrangian is

$$\hat{L} = \frac{\hat{p}^2}{2m} - V(\hat{x}),$$

we find the Hamiltonian to be

$$\begin{aligned}\hat{H} &= \hat{p}\dot{\hat{x}} - \hat{L}\\ &= \hat{p}\left(\frac{\hat{p}}{m}\right) - \left(\frac{\hat{p}^2}{2m} - V(\hat{x})\right)\\ &= \frac{\hat{p}^2}{2m} + V(\hat{x}),\end{aligned}$$

and to make the calculation of the spectrum simpler, we assume that $V(x)$ is identically equal to 0. The spectrum, in this case then, turns out to be consisting of all positive reals; that is, any positive real number gives a possible value for the energy of the particle. To clarify this point further, we note that with $V(x)$ set equal to 0, the particle is not confined in any manner whatsoever, something meaning that its state could be the momentum eigenstate $\left(1/\sqrt{2\pi}\right)\exp(\imath px)$ with $p \in \mathbb{R}$. Then, since the Hamiltonian is $\hat{p}^2/2m$, we know from ordinary quantum mechanics that the energy is $p^2/2m$, which could be any positive real given that p is arbitrary. This thing completes application of the canonical quantization program to the system of the point particle. As to its applications to fields, we present them in the next section.

1.3 Klein-Gordon and Dirac fields

Being a (free) scalar field, the *Klein-Gordon field* (or *KG field* for short) is the simplest to be studied both classically and quantum mechanically; therefore, it is the first field that we choose to demonstrate the full application of canonical quantization to. The Lagrangian density governing the KG field is

$$\mathcal{L} = \frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2, \tag{1.10}$$

which we could use to calculate the conjugate field $\pi = \partial\mathcal{L}/\partial\dot{\varphi} = \dot{\varphi}$. Then, going with the prescription of the canonical quantization program, we promote φ and π to operators and impose the equal-time commutation relations in Eq. (1.9). Promotion of φ and π to operators renders the Hamiltonian an operator as well, and we are thus allowed to write

$$\hat{H} = \int d^3\mathbf{x} \left(\frac{1}{2}\hat{\pi}^2 + \frac{1}{2}\left(\nabla\hat{\varphi}\right)^2 + \frac{1}{2}m^2\hat{\varphi}^2 \right). \tag{1.11}$$

As the final step of the scheme, we need to calculate the spectrum of the Hamiltonian, but since we are dealing with infinitely many degrees of freedom—rather than the finitely many seen in the case of the infinitesimal particle—finding the spectrum is very hard. We thus begin by seeking guidance from the equation of motion of the KG field written in the Fourier space. The equation of motion in the position space, also known as the KG equation, could be found by using the Lagrangian density in Eq. (1.10) in conjunction with the Euler-Lagrange equation, which yields

$$(\partial^\mu\partial_\mu + m^2)\varphi = 0. \tag{1.12}$$

Then, expanding (or Fourier transforming) the classical KG field as

$$\varphi(t,\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} e^{i\mathbf{p}\cdot\mathbf{x}} \varphi(t,\mathbf{p})$$

(with $\varphi(t,\mathbf{p})^* = \varphi(t,-\mathbf{p})$ so that $\varphi(t,\mathbf{x})$ is real), we find that the KG equation becomes

$$\left[\frac{\partial^2}{\partial t^2} + \left(|\mathbf{p}|^2 + m^2\right)\right]\varphi(t,\mathbf{p}) = 0,$$

which implies that the solution is something of a harmonic oscillator with frequency $\omega_{\mathbf{p}}$ such that $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$. Inspired by this observation, we try to write the spectrum of the KG-field Hamiltonian in terms of the spectrum of a harmonic oscillator, which we are well-acquainted with. An important part of the calculation of this spectrum involves the creation and annihilation operators, and for a harmonic oscillator described by the Hamiltonian

$$\hat{H}_{\text{SHO}} = \frac{1}{2}\hat{p}^2 + \frac{1}{2}\omega^2\hat{\varphi}^2,$$

they are introduced via

$$\hat{\varphi} = \frac{1}{\sqrt{2\omega}}\left(\hat{a} + \hat{a}^\dagger\right)$$

and

$$\hat{p} = -\iota\sqrt{\frac{\omega}{2}}\left(\hat{a} - \hat{a}^\dagger\right).$$

with $\left[\hat{a}, \hat{a}^\dagger\right] = 1$. Using these very relations and a simple comparison with the Fourier transform of the classical KG field above, we could write

$$\hat{\varphi}(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right)e^{\iota\mathbf{p}\cdot\mathbf{x}}$$

and

$$\hat{\pi}(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}(-\iota)\sqrt{\frac{\omega_{\mathbf{p}}}{2}}\left(\hat{a}_{\mathbf{p}} - \hat{a}^\dagger_{-\mathbf{p}}\right)e^{\iota\mathbf{p}\cdot\mathbf{x}},$$

which essentially say that each mode of the field is an independent harmonic oscillator with its own creation and annihilation operators. We see that if $\left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{\mathbf{p}'}\right] = (2\pi)^3\delta^{(3)}(\mathbf{p} - \mathbf{p}')$, then these *mode* *expansions* for $\hat{\varphi}$ and $\hat{\pi}$ allow the commutation relation $[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')]$ to be worked out correctly:

$$\begin{aligned}\left[\hat{\varphi}(\mathbf{x}), \hat{\pi}(\mathbf{x}')\right] &= \int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}\frac{-\imath}{2}\sqrt{\frac{\omega_{\mathbf{p}'}}{\omega_{\mathbf{p}}}}\left(\left[\hat{a}^\dagger_{-\mathbf{p}}, \hat{a}_{\mathbf{p}'}\right] - \left[\hat{a}_{\mathbf{p}}, \hat{a}^\dagger_{-\mathbf{p}'}\right]\right)e^{\imath(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}')} \\ &= \imath\delta^{(3)}(\mathbf{x}-\mathbf{x}').\end{aligned}$$

An important thing to note is that we have started working in the Schrodinger picture here. This thing is essential since we wrote the mode expansions by way of a comparison with the expressions for the harmonic oscillator, all of which were written in the Schrodinger picture. However, we could shift to the Heisenberg picture any time once we have found the Hamiltonian in terms of the creation and annihilation operators, for $\hat{a}_{\mathbf{p}}(t) = \exp\left(\imath\hat{H}t\right)\hat{a}_{\mathbf{p}}\exp\left(-\imath\hat{H}t\right)$ and $\hat{a}^\dagger_{\mathbf{p}}(t) = \exp\left(\imath\hat{H}t\right)\hat{a}^\dagger_{\mathbf{p}}\exp\left(-\imath\hat{H}t\right)$. Using the mode expansions for $\hat{\varphi}$ and $\hat{\pi}$ in Eq. (1.11), we see that the Hamiltonian turns out to be

$$\begin{aligned}\hat{H} = &-\int d^3\mathbf{x}\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}e^{\imath(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}')}\frac{\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}{4}\left(\hat{a}_{\mathbf{p}} - \hat{a}^\dagger_{-\mathbf{p}}\right)\left(\hat{a}_{\mathbf{p}'} - \hat{a}^\dagger_{-\mathbf{p}'}\right) \\ &+\int d^3\mathbf{x}\int \frac{d^3\mathbf{p}d^3\mathbf{p}'}{(2\pi)^6}e^{\imath(\mathbf{p}\cdot\mathbf{x}+\mathbf{p}'\cdot\mathbf{x}')}\frac{-\mathbf{p}\cdot\mathbf{p}'+m^2}{4\sqrt{\omega_{\mathbf{p}}\omega_{\mathbf{p}'}}}\left(\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right)\left(\hat{a}_{\mathbf{p}'} + \hat{a}^\dagger_{-\mathbf{p}'}\right) \\ = &\int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_{\mathbf{p}}\left(\hat{a}^\dagger_{\mathbf{p}}\hat{a}_{\mathbf{p}} + \frac{1}{2}(2\pi)^3\delta^{(3)}(0)\right).\end{aligned}$$

The appearance of $\delta^{(3)}(0)$ is to be expected since the second term represents the sum of the zero-point energies of all modes, and we are dealing with an infinite number of them here. However, since this infinite energy shift cannot be detected experimentally—all experiments measuring energy differences only—we could ignore this infinite term in all our calculations.

With the Hamiltonian found, we could show that $\left[\hat{H}, \hat{a}^\dagger_{\mathbf{p}}\right] = \omega_{\mathbf{p}}\hat{a}^\dagger_{\mathbf{p}}$ and $\left[\hat{H}, \hat{a}_{\mathbf{p}}\right] = -\omega_{\mathbf{p}}\hat{a}_{\mathbf{p}}$, something which allows us to write down the spectrum for the Hamiltonian just as for the harmonic oscillator. Since $\hat{a}_{\mathbf{p}}\left|0\right\rangle = 0$ for all $\mathbf{p}$, it is taken as the ground state with an energy of 0 after our dropping the infinite constant term above. All other energy eigenstates could be obtained by acting on $|0\rangle$ with creation operators. In general, $a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} \ldots |0\rangle$ is the eigenstate of $\hat{H}$ with energy $\omega_{\mathbf{p}} + \omega_{\mathbf{q}} + \ldots$. This thing completes the implementation of canonical quantization to the KG field, but at the same time, it presents us with a particularly important conclusion, something which has to do with the statistics of the states above.

As we stated in section 1.1, the conserved charge corresponding to spacetime translation symmetry turns out to be four-momentum, and with the expressions we derived for it and the energy-momentum tensor, we could find the three-momentum carried by the field:

$$\mathbf{P} = -\int d^3\mathbf{x} \pi(\mathbf{x}) \nabla \varphi(\mathbf{x}).$$

Then, promoting φ and π to operators, using their respective mode expansions, and performing a calculation similar to the one done in the case of the Hamiltonian, we find that

$$\hat{\mathbf{P}} = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \mathbf{p} \hat{a}^\dagger_{\mathbf{p}} \hat{a}_{\mathbf{p}},$$

which tells us that $a^\dagger_{\mathbf{p}}$ creates momentum $\mathbf{p}$ and (as we saw for the KG equation in the Fourier space) energy $\omega_{\mathbf{p}} = \sqrt{|\mathbf{p}|^2 + m^2}$. Likewise, the state $a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{q}} \ldots |0\rangle$ could be interpreted as having momentum $\mathbf{p} + \mathbf{q} + \ldots$. Naturally, one feels inclined to call these *excitations* particles, for they are discrete entities obeying the relativistic energy-momentum relation, but it must be realized that a particle here is an excitation in some momentum eigenstate, not an entity localized in space. We do discuss how the KG field creates particles in the position space later, but the important thing for now is that the formalism we have developed allows us to determine the statistics of our momentum eigenstates. Considering the two-particle state $\hat{a}^\dagger_{\mathbf{p}} \hat{a}^\dagger_{\mathbf{q}} |0\rangle$, we see that it is identical to the state $\hat{a}^\dagger_{\mathbf{q}} \hat{a}^\dagger_{\mathbf{p}} |0\rangle$, which has the two particles interchanged. Moreover, a single momentum mode could have arbitrarily many particles ($a^\dagger_{\mathbf{p}} a^\dagger_{\mathbf{p}} \ldots |0\rangle$). We are thus led to the conclusion that these KG particles obey the *Bose-*

Einstein statistics and could hence be reconginzed as *bosons*. In the wake of this observation, the KG field itself is referred to as a *bosonic field*.

Having then learnt the way to (canonically) quantize a field related to bosons, we also note that there exist particles that do not play by the same rules. Referred to as being *fermions*, they could definitely not be related to a field like the KG field. What is needed is a field that gives rise to the *Fermi-Dirac statistics* rather than the Bose-Einstein ones, and one such field happens to be the so-called *Dirac field*. A field of mathematical entities known as *spinors*, the Dirac field is governed by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(\imath\gamma^{\mu}\partial_{\mu} - m)\psi, \tag{1.13}$$

where ψ is a spinor, or a 4×1 column vector, the γ^{μ} are some 4×4 matrices that satisfy $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta_{\mu\nu}$, and $\bar{\psi}$ is a row vector defined as $\psi^{\dagger}\gamma^{0}$. With these definitions, we see that $\mathcal{L}$ is a scalar, but as to what motivates such definitions, we leave it out in the interest of covering the essentials of quantizing the Dirac field through the canonical quantization program.

Starting in the usual way, we use $\mathcal{L}$ to calculate the field conjugate to ψ: $\pi = \partial\mathcal{L}/\partial\dot{\psi} = \imath\bar{\psi}\gamma^{0} = \imath\psi^{\dagger}$, where we employed $\{\gamma^{\mu}, \gamma^{\nu}\} = 2\eta_{\mu\nu}$ and $\bar{\psi} = \psi^{\dagger}\gamma^{0}$ for the last equality. Then, as before, we promote ψ and $\psi^{\dagger}$ to operators, thereby reaching the point that commutation relations are to be imposed at. This very point is the one that the canonical quantization of the Dirac field departs at from that of the KG field. It turns out that in order to have the right scheme for fields related to fermions, one needs to use anticommutation relations in lieu of the commutation ones, and once again, we leave the detailed discussion of this point out in the interest of covering the essentials of the quantization. The anticommutation relations we impose are the following, and like equal-time commutation relations, they too are defined for equal times.

$$\left\{\hat{\psi}_\alpha(t,\mathbf{x}),\hat{\psi}_\beta(t,\mathbf{y})\right\} = \left\{\hat{\psi}^\dagger_\alpha(t,\mathbf{x}),\hat{\psi}^\dagger_\beta(t,\mathbf{y})\right\} = 0;$$
$$\left\{\hat{\psi}_\alpha(t,\mathbf{x}),\hat{\psi}^\dagger_\beta(t,\mathbf{y})\right\} = \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}). \tag{1.14}$$

Here, α and β represent what are called *spinor indices*, or the components of the spinors being used in the anticommutation relations. Once again, promoting ψ and $\psi^\dagger$ to operators turns the Hamiltonian into an operator as well, and we could write

$$\hat{H} = \int d^3\mathbf{x}\left(\hat{\pi}(\mathbf{x})\hat{\varphi}(\mathbf{x}) - \hat{\mathcal{L}}\right) = \int d^3\mathbf{x}\bar{\hat{\psi}}(-\imath\gamma^i\partial_i + m)\hat{\psi}. \tag{1.15}$$

Just like the KG field, the Dirac field also presents us with infinitely many degrees of freedom, making it extremely difficult to determine the spectrum of the Hamiltonian; therefore, we again resort to some mode expansions to get around this thing:

$$\hat{\psi}(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}e^{\imath\mathbf{p}\cdot\mathbf{x}}\sum_{s=1}^{2}\left(\hat{a}^s_\mathbf{p}u^s(\mathbf{p}) + \hat{b}^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p})\right)$$

and

$$\hat{\psi}^\dagger(\mathbf{x}) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_\mathbf{p}}}e^{\imath\mathbf{p}\cdot\mathbf{x}}\sum_{s=1}^{2}\left(\hat{b}^s_\mathbf{p}v^{s\dagger}(\mathbf{p}) + \hat{a}^{s\dagger}_{-\mathbf{p}}u^{s\dagger}(-\mathbf{p})\right),$$

where $\hat{a}^\dagger_\mathbf{p}$ and $\hat{b}^\dagger_\mathbf{p}$ are creation operators, $\hat{a}_\mathbf{p}$ and $\hat{b}_\mathbf{p}$ are annihilation operators, and $u^s(\mathbf{p})$ and $v^s(\mathbf{p})$ are some spinors. Even though we do not put forth the rationale for these mode expansions, we do notice their resemblance with the mode expansions for the KG field, something which hints at their having been obtained via a similar procedure. If $\left\{\hat{a}^r_\mathbf{p},\hat{a}^{s\dagger}_\mathbf{q}\right\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ and $\left\{\hat{b}^r_\mathbf{p},\hat{b}^{s\dagger}_\mathbf{q}\right\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q})$ now, with all the other anticommutators vanishing, then $\left\{\hat{\psi}_\alpha(\mathbf{x}),\hat{\psi}^\dagger_\beta(\mathbf{y})\right\}$ could be worked out correctly:

$$
\begin{aligned}
\left\{\hat{\psi}_\alpha(\mathbf{x}), \hat{\psi}^\dagger_\beta(\mathbf{y})\right\} &= \int \frac{d^3\mathbf{p}d^3\mathbf{q}}{(2\pi)^6}\frac{1}{\sqrt{4\omega_\mathbf{p}\omega_\mathbf{q}}} \\
&\qquad \times \sum_{r,s}\left(\left\{\hat{a}^r_\mathbf{p}, \hat{a}^{s\dagger}_\mathbf{q}\right\} u^r_\alpha(\mathbf{p})u^{s\dagger}_\beta(\mathbf{q}) + \left\{\hat{b}^r_{-\mathbf{p}}, \hat{b}^{s\dagger}_{-\mathbf{q}}\right\} v^r_\alpha(-\mathbf{p})v^{s\dagger}_\beta(-\mathbf{q})\right) \\
&\qquad \times e^{\imath(\mathbf{p}\cdot\mathbf{x}+\mathbf{q}\cdot\mathbf{y})} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_\mathbf{p}}e^{\imath\mathbf{p}\cdot(\mathbf{x}-\mathbf{y})} \\
&\qquad \times \left[\left(\gamma^0\omega_\mathbf{p} - \boldsymbol{\gamma}\cdot\mathbf{p} + m\right)_{\alpha\beta} + \left(\gamma^0\omega_\mathbf{p} + \boldsymbol{\gamma}\cdot\mathbf{p} - m\right)_{\alpha\beta}\right]\gamma^0 \\
&= \delta_{\alpha\beta}\delta^{(3)}(\mathbf{x}-\mathbf{y}),
\end{aligned}
$$

where we used some relations satisfied by the $u^s(\mathbf{p})$ and the $v^s(\mathbf{p})$ for the first and second equalities: $\sum_s u^s(\mathbf{p})\bar{u}^s(\mathbf{p}) = \gamma^\mu p_\mu + m$, $\sum_s v^s(\mathbf{p})\bar{v}^s(\mathbf{p}) = \gamma^\mu p_\mu - m$, and $u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0$. Just as in the case of the KG field, we notice that we are working in the Schrodinger picture here. We could, however, transition to the Heisenberg picture once we have the Hamiltonian in terms of the creation and annihilation operators. To obtain such an expression for the Hamiltonian, we use our mode expansions for ψ and $\psi^\dagger$ in conjunction with Eq. (1.15), thereby getting the following:

$$
\begin{aligned}
\hat{H} &= \int d^3\mathbf{x}\hat{\psi}^\dagger\gamma^0\left(-\imath\gamma^i\partial_i + m\right)\hat{\psi} \\
&= \int \frac{d^3\mathbf{x}d^3\mathbf{p}d^3\mathbf{q}}{(2\pi)^6}\sqrt{\frac{\omega_\mathbf{p}}{4\omega_\mathbf{q}}}\sum_{r,s}\left(\hat{b}^r_\mathbf{q}v^{r\dagger}(\mathbf{q}) + \hat{a}^{r\dagger}_{-\mathbf{q}}u^{r\dagger}(-\mathbf{q})\right)\left(\hat{a}^s_\mathbf{p}u^s(\mathbf{p}) - \hat{b}^{s\dagger}_{-\mathbf{p}}v^s(-\mathbf{p})\right)e^{\imath(\mathbf{p}+\mathbf{q})\cdot\mathbf{x}} \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2}\sum_{r,s}\left[\hat{a}^{r\dagger}_\mathbf{p}\hat{a}^s_\mathbf{p}\left(u^{r\dagger}(\mathbf{p})u^s(\mathbf{p})\right) - \hat{b}^r_\mathbf{p}\hat{b}^{s\dagger}_\mathbf{p}\left(v^{r\dagger}(\mathbf{p})v^s(\mathbf{p})\right)\right. \\
&\qquad\qquad \left. - \hat{a}^{r\dagger}_\mathbf{p}\hat{b}^{\dagger s}_\mathbf{p}\left(u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p})\right) + \hat{b}^r_\mathbf{p}\hat{a}^s_{-\mathbf{p}}\left(v^{r\dagger}(\mathbf{p})u^s(-\mathbf{p})\right)\right] \\
&= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_\mathbf{p}\sum_s\left(\hat{a}^{s\dagger}_\mathbf{p}\hat{a}^s_\mathbf{p} - \hat{b}^s_\mathbf{p}\hat{b}^{s\dagger}_\mathbf{p}\right) = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\omega_\mathbf{p}\sum_s\left(\hat{a}^{s\dagger}_\mathbf{p}\hat{a}^s_\mathbf{p} + \hat{b}^{s\dagger}_\mathbf{p}\hat{b}^s_\mathbf{p} - (2\pi)^3\delta^{(3)}(0)\right),
\end{aligned}
$$

where we used $\left(-\gamma^i p_i + m\right) u^s(\mathbf{p}) = \gamma^0 p_0 u^s(\mathbf{p})$ and $\left(\gamma^i p_i + m\right) v^s(\mathbf{p}) = -\gamma^0 p_0 v^s(\mathbf{p})$ for the second equality and $u^{r\dagger}(\mathbf{p})u^s(\mathbf{p}) = v^{r\dagger}(\mathbf{p})v^s(\mathbf{p}) = 2\omega_{\mathbf{p}}\delta^{rs}$ and $u^{r\dagger}(\mathbf{p})v^s(-\mathbf{p}) = v^{r\dagger}(-\mathbf{p})u^s(\mathbf{p}) = 0$ for the penultimate one. Now, since all experiments measure energy differences, the infinite energy shift $-(2\pi)^3\delta^{(3)}(0)$ cannot be detected experimentally, and we could ignore it in all calculations. What really needs to be noted here is that we did the final step, which involves the use of the anticommutation relation $\left\{\hat{a}^r_{\mathbf{p}}, \hat{a}^{s\dagger}_{\mathbf{q}}\right\} = (2\pi)^3\delta^{rs}\delta^{(3)}(\mathbf{p}-\mathbf{q})$, explicitly this time. It clearly shows that if we had a commutation relation instead, then the integrand in the Hamiltonian would end up having the term $-\omega_{\mathbf{p}}\hat{b}^{s\dagger}_{\mathbf{p}}\hat{b}^s_{\mathbf{p}}$, which, in analogy with the particle interpretation we presented in the case of the KG field, tells us that a particle created by the operator $\hat{b}^{s\dagger}_{\mathbf{p}}$ corresponds to the negative energy $-\omega_{\mathbf{p}}$. That is, by creating more and more particles with $\hat{b}^{s\dagger}_{\mathbf{p}}$, we could lower the energy indefinitely, something which is disturbingly unphysical. This fact explains why canonical quantization applied to *fermionic fields* (like the Dirac field) employs anticommutation relations rather than the commutation ones.

With the Hamiltonian in hand, we could easily show that $\left[\hat{H}, \hat{a}^{s\dagger}_{\mathbf{p}}\right] = \omega_{\mathbf{p}}\hat{a}^{s\dagger}_{\mathbf{p}}$, $\left[\hat{H}, \hat{a}^s_{\mathbf{p}}\right] = -\omega_{\mathbf{p}}\hat{a}^s_{\mathbf{p}}$, $\left[\hat{H}, \hat{b}^{s\dagger}_{\mathbf{p}}\right] = \omega_{\mathbf{p}}\hat{b}^{s\dagger}_{\mathbf{p}}$, and $\left[\hat{H}, \hat{b}^s_{\mathbf{p}}\right] = -\omega_{\mathbf{p}}\hat{b}^s_{\mathbf{p}}$. Then, if we let $|0\rangle$ be the ground state, that is, $\hat{a}^s_{\mathbf{p}}|0\rangle = \hat{b}^s_{\mathbf{p}}|0\rangle = 0$ for all $\mathbf{p}$, we could obtain all the other energy eigenstates by acting on $|0\rangle$ with the operators $\hat{a}^{s\dagger}_{\mathbf{p}}$ and $\hat{b}^{s\dagger}_{\mathbf{p}}$. This thing exhausts the spectrum of the Hamiltonian and completes the application of the canonical quantization scheme to the Dirac field. As to the statistics of the states obtained in this case, we could say that they are the Fermi-Dirac statistics. Exchange of two particles yields a negative sign: $|\mathbf{p}, r; \mathbf{q}, s\rangle = \hat{a}^{r\dagger}_{\mathbf{p}}\hat{a}^{s\dagger}_{\mathbf{q}}|0\rangle = -\hat{a}^{s\dagger}_{\mathbf{q}}\hat{a}^{r\dagger}_{\mathbf{p}}|0\rangle = -|\mathbf{q}, s; \mathbf{p}, r\rangle$. Moreover, no two particles could occupy the same state (also knwon as the *Pauli exclusion principle*): $|\mathbf{p}, s; \mathbf{p}, s\rangle = -|\mathbf{p}, s; \mathbf{p}, s\rangle = 0$. One last thing we note is that the canonical quantization of the Dirac field yields two kinds of operators: the a's and the b's. Whereas the former deal with the good old particles, the latter correspond to what are known as *antiparticles*, but we leave any further discussion of this specific topic, for we have attained the goal this section was primarily meant for: demonstrating the application of canonical quantization to fields.

In addition to describing how fields—both bosonic and fermionic—are to be quantized canonically, we use this section to make precise a claim we made in chapter 0, namely that a quantum field is capable of creating particles at any spacetime point. We illustrate this point via the example of the KG field. Now that we have quantized it, we have obtained a quantum field, $\hat{\varphi}(\mathbf{x})$, from the classical KG field, $\varphi(\mathbf{x})$, and we claim that $\hat{\varphi}(\mathbf{x})$ is capable of creating a particle in the vacuum at position $\mathbf{x}$. Explicitly,

$$\begin{aligned}\hat{\varphi}(\mathbf{x})|0\rangle &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right)e^{\imath\mathbf{p}\cdot\mathbf{x}}|0\rangle \\ &= \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}e^{\imath\mathbf{p}\cdot\mathbf{x}}|-\mathbf{p}\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2\omega_{\mathbf{p}}}e^{-\imath\mathbf{p}\cdot\mathbf{x}}|\mathbf{p}\rangle\,,\end{aligned}$$

where we have used that $|-\mathbf{p}\rangle = \sqrt{2\omega_{-\mathbf{p}}}\hat{a}^\dagger_{-\mathbf{p}}|0\rangle$ (which comes from the Lorentz invariant normalization of the momentum eigenstates) and $\omega_{-\mathbf{p}} = \omega_{\mathbf{p}}$ for the second equality. This expression, apart from the factor $1/2\omega_{\mathbf{p}}$, is the same as the nonrelativistic expression for the eigenstate $|\mathbf{x}\rangle$; in fact, the extra factor is almost constant for nonrelativistic $\mathbf{p}$. Therefore, we could say that $\hat{\varphi}(\mathbf{x})$ creates a particle in the vacuum at position $\mathbf{x}$. This thing becomes further apparent with the following calculation:

$$\langle 0|\hat{\varphi}(\mathbf{x})|\mathbf{p}\rangle = \langle 0|\int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{\sqrt{2\omega_{\mathbf{p}}}}\left(\hat{a}_{\mathbf{p}} + \hat{a}^\dagger_{-\mathbf{p}}\right)e^{\imath\mathbf{p}\cdot\mathbf{x}}\sqrt{2\omega_{\mathbf{p}}}\hat{a}^\dagger_{\mathbf{p}}|0\rangle = e^{\imath\mathbf{p}\cdot\mathbf{x}} \propto \langle\mathbf{x}|\mathbf{p}\rangle\,.$$

1.4 Path integral formalism

Up till now, we have covered only one route to quantization, namely the canonical quantization. This section aims to present an alternate route known as the *path integral formalism*. Not only does it have the advantage of being much simpler, but it also is quite intuitive, for it uses classical quantities in lieu of quantum ones, something implying that one does not get into the hassle of dealing with operators and commutation relations.

Our approach involves using the canonical quantization to derive the path integration method for the simple classical system we introduced in section 1.2: a point particle of mass m moving in potential $V(x)$. The simple derivation we do later helps us to generalize our method to the context of fields. With canonical quantization, or promotion of classical degrees of freedom to operators, we could write the Hamiltonian for the system we wish to consider as

$$\hat{H} = \frac{\hat{p}^2}{2m} + V(\hat{x}), \tag{1.16}$$

where $\hat{x}$ and $\hat{p}$, as before, denote the position and momentum operators, respectively, and satisfy the commutation relation $[\hat{x}, \hat{p}] = \imath$. The primary quantum mechanics problem for this system is the calculation of the amplitude that the particle starts in the state $|x_i\rangle$ and ends in the state $|x_f\rangle$ after time t, and ordinary quantum mechanics does this calculation via the unitary time-evolution operator $\hat{U}(t) = \exp\left(-\imath\hat{H}t\right)$, which is a solution to the time-dependent Schrodinger equation. In other words, ordinary quantum mechanics computes $\langle x_f|\, \hat{U}(t)\, |x_i\rangle$, which could be viewed simply as the overlap of the time-evolved initial state $\left(\hat{U}(t)\,|x_i\rangle\right)$ with the final state $\left(|x_f\rangle\right)$. We show how this overlap could be written as a *path integral*, but to do so, we need to compute $\langle x|\, \hat{U}(\partial t)\, |x'\rangle$ to first order in ∂t, where ∂t is a small time interval:

$$\begin{aligned}
\langle x|\, e^{-\imath\left(\hat{p}^2/2m+V(\hat{x})\right)\partial t}\, |x'\rangle &= \langle x|\, e^{-\imath\left(\hat{p}^2/2m\right)\partial t} e^{-\imath V(\hat{x})\partial t} e^{O((\partial t)^2)}\, |x'\rangle \\
&\approx \int \frac{dp}{2\pi}\, \langle x|\, e^{-\imath\left(\hat{p}^2/2m\right)\partial t}\, |p\rangle \, \langle p|\, e^{-\imath V(\hat{x})\partial t}\, |x'\rangle \\
&= \int \frac{dp}{2\pi} \exp\left[-\imath\partial t \left(\frac{p^2}{2m} - p\frac{(x-x')}{\partial t}\right) + V(x')\right] \\
&= \sqrt{\frac{m}{2\pi\imath\partial t}} \exp\left[\imath\partial t\left(\frac{1}{2}m\frac{(x-x')^2}{\partial t^2} - V(x')\right)\right].
\end{aligned}$$

The quantity in the argument of the final exponential—barring the $\imath$—is nothing but the infinitesimal action $S(x', x; \partial t)$, which corresponds to the particle's moving from x' to x in time ∂t; therefore, to first order in ∂t, we may write

$$\langle x| \hat{U}(\partial t) |x'\rangle = \sqrt{\frac{m}{2\pi\iota\partial t}} \exp\big(\iota S(x', x; \partial t)\big).$$

With this result in hand, we could go on to write $\langle x_f| \hat{U}(t) |x_i\rangle$ in a similar manner. All we need to do is to divide t into N subintervals, each of duration t/N, insert appropriate completeness relations, and apply a limiting procedure:

$$\begin{aligned}
\langle x_f| \hat{U}(t) |x_i\rangle &= \langle x_f| \underbrace{\hat{U}(t/N) \dots \hat{U}(t/N)}_{N \text{ terms}} |x_i\rangle \\
&= \int \prod_{j=1}^{N-1} dx_j \, \langle x_f| \hat{U}(t/N) |x_{N-1}\rangle \dots \langle x_1| \hat{U}(t/N) |x_i\rangle \\
&= \lim_{N\to\infty} \left(\frac{mN}{2\pi\iota t}\right)^{N/2} \int \prod_{j=1}^{N-1} dx_j \exp(\iota S[x]) \\
&= \int_{(x_i,0)}^{(x_f,t)} [dx] \exp(\iota S[x]),
\end{aligned} \tag{1.17}$$

where $[dx] = \lim_{N\to\infty} \prod_{j=1}^{N-1} \left(\sqrt{mN/2\pi\iota t}dx_j\right)$, $S[x] = \int dt\mathcal{L}(x, \dot{x})$, and $\mathcal{L}(x, \dot{x}) = (1/2)m\dot{x}^2 - V(x)$. The expression in the last line is what we have been referring to as a path integral, and it gives an alternate way to calculate the amplitude $\langle x_f| \hat{U}(t) |x_i\rangle$, a way which does not entail having to deal with quantum states, operators, and so on; rather, a classical quantity, the action $S[x]$, is used for the calculation. The interpretation of Eq. (1.17) is that every possible trajectory going from x_i to x_f in time t contributes to the amplitude with a weight equal to the exponential of ι times the corresponding action. The trajectories that action varies the least around (also known as classical trajectories) contribute the most; the others get their contributions more or less canceled due to the huge variation of action in the vicinity and the weight's being the oscillating exponential $\exp(\iota S[x])$.

At any rate, summing all contributions in the limit that their number gets infinitely large, or integrating $\exp(\iota S[x])$ over all possible *paths* going from x_i to x_f in time t, yields the full amplitude $\langle x_f | \hat{U}(t) | x_i \rangle$. We have clearly found a new way to compute the said amplitude, and it not only bypasses the machinery of ordinary quantum mechanics but also allows us to use our knowledge of classical mechanics to do the calculation. In fact, this new scheme could be used to calculate the amplitude for the general state $|\psi_i\rangle$ to evolve into the state $|\psi_f\rangle$ after time t since

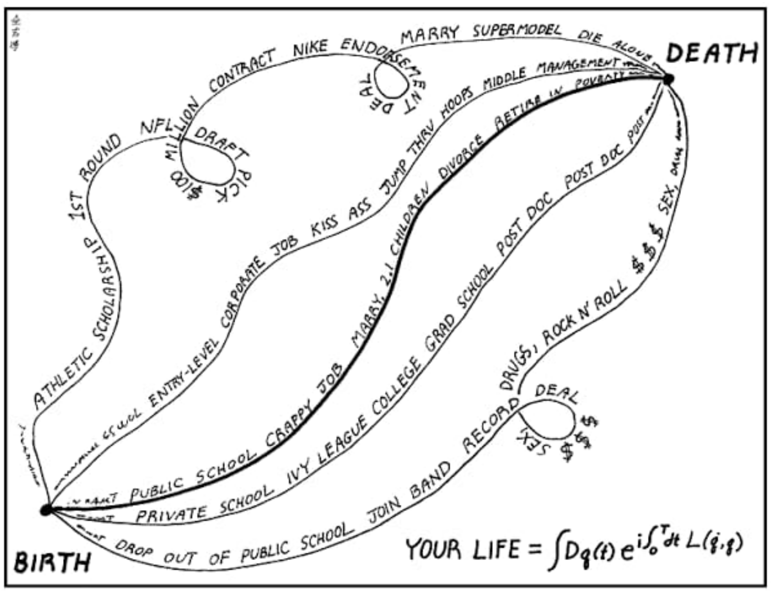
$$\langle \psi_f | \hat{U}(t) | \psi_i \rangle = \int dx_i dx_j \psi_f^*(x_f) \psi_i(x_i) \langle x_f | \hat{U}(t) | x_i \rangle ,$$

and we already have the path integral expression for $\langle x_f | \hat{U}(t) | x_i \rangle$. This very thing provides us with the motivation to see Eq. (1.17) as the starting point for all of quantum mechanics. It is completely equivalent to the Schrodinger equation (in terms of incorporating system dynamics) and allows for the calculation of the same quantities—albeit in an alternate manner. We have derived it for a time-independent Hamiltonian, but the result holds for a time-dependent Hamiltonian as well since all that matters is the infinitesimal amplitude $\langle x | \hat{U}(\delta t) | x' \rangle$.

With this simple derivation of the path integral formalism done, we state that the path integral quantization of a bosonic field, φ, is not more difficult conceptually. What we do is simply redefine the integration measure by dividing the spacetime into infinitesimal intervals and integrating over $\varphi(t, \mathbf{x})$ at every $\mathbf{x}$. The probability amplitude for the transition of the configuration $\varphi_i(t_i, \mathbf{x})$ into the configuration $\varphi_f(t_f, \mathbf{x})$ may then be written as

$$\left\langle \varphi_f(t_f, \mathbf{x}) \middle| \varphi_i(t_i, \mathbf{x}) \right\rangle = \int [d\varphi(t, \mathbf{x})] \exp(\iota S[\varphi]). \tag{1.18}$$

A similar path integral could be written for a fermionic field as well, but since doing so requires introducing objects called Grassmann numbers and related paraphernalia, we defer writing it (or something closely related to it to be precise) to a future point in this work. Having introduced classical fields and the techniques to transition to quantum ones in this chapter, we move to actually using quantum field theory to calculate physically important quantities in the next. In particular, we calculate quantities called correlators, which are crucial as far as processes like scattering are concerned. This calculation forms the subject matter of the next chapter primarily, and we do it via both of the quantization techniques we introduced in this chapter—canonical quantization and path integration.



The Path Integral Formulation of Your Life

Figure 1.2: Path integral formalism. Despite being a little facetious, this analogy captures the spirit of the path integrals approach. Although all paths contribute, the contributions of the most deviant ones cancel.

"Since then I never pay attention to anything by 'experts.'
I calculate everything myself."

Richard Feynman

2

Correlators[5,7]

Now that we have erected a mathematical scaffolding to work with quantum fields and quantum field theory, we could start doing some simple but important calculations. They are simple in that they involve what are known as free fields only—fields not interacting with themselves or any other field—and they are important in that they bear physical import of a great consequence. To be precise, we calculate quantities known as correlators, which, loosely speaking, are like the amplitudes we discussed in section 1.4 of the previous chapter. They prove quite significant when it comes to calculating probabilities, cross sections, and so on for various kinds of scattering processes, an example being two incoming particles' getting destroyed to produce two outgoing (or scattered) particles. We start by delineating the meaning of a correlator and discussing the mathematical preliminaries that allow us to calculate correlators via both Hamiltonian field theory and path integral formalism. Then, we explicitly do these calculations for the Klein-Gordon and Dirac fields, working out extremely simple correlators for each of them. Finally, we append a short section detailing the derivation of the so-called Wick's theorem through the mathematical techniques we introduce right in the beginning of this chapter.

2.1 Mathematical preamble

Quantum field theory deals with transition (or scattering) amplitudes between asymptotic states, also known as free particles, and these amplitudes are given by correlation functions, or correlators. Given a field, φ, we denote its n-point correlator as $\langle\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\rangle$ and define it as

$$\langle\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\rangle = \langle 0|\,\mathcal{T}\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\right)|0\rangle\,, \tag{2.1}$$

where $|0\rangle$ is the ground state and $\mathcal{T}$ is the time ordering operator. This operator puts the factors following it (which happen to be operators themselves) in a chronological order from right to left; that is,

$$\begin{aligned}
&\mathcal{T}\left(\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\right)\\
&=\mathcal{T}\left(\hat{\varphi}(t_1,\mathbf{x}_1)\hat{\varphi}(t_2,\mathbf{x}_2)\ldots\hat{\varphi}(t_n,\mathbf{x_n})\right)\\
&=\hat{\varphi}(t_1,\mathbf{x}_1)\hat{\varphi}(t_2,\mathbf{x}_2)\ldots\hat{\varphi}(t_n,\mathbf{x_n})\\
&=\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)
\end{aligned}$$

if $t_1 > t_2 > \ldots > t_n$. Just looking at the expression in Eq. (2.1) and recalling the mode expansions covered in section 1.3 tell us how we could go about evaluating a correlator with the Hamiltonian field theory: simply replace the field operators with their corresponding mode expansions, collect the nonzero terms after the action of the creation and annihilation operators, and perform appropriate integrals. What needs to be considered, however, is that the same correlator could also be evaluated with the path integral formalism, and we discuss the details of the calculation here. In order to keep the discussion simple, we follow precisely the same sequence as the one we adopted to first introduce path integrals in chapter 1; that is, we begin by presenting the problem in the context of a point particle rather than fields.

Keeping the comparisons between the point particle and fields in view, we may write the n-point correlator for the former as

$$\langle \hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n)\rangle = \langle 0|\,\mathcal{T}\left(\hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n)\right)|0\rangle\,.$$

This expression could be manipulated and written instead as

$$\frac{\langle 0|\,\hat{x}e^{\imath\hat{H}(t_2-t_1)}\hat{x}e^{\imath\hat{H}(t_3-t_2)}\ldots\hat{x}\,|0\rangle}{\langle 0|\,e^{\imath\hat{H}(t_n-t_1)}\,|0\rangle},$$

for from ordinary quantum mechanics, we know that $\hat{x}(t) = \exp\left(\imath\hat{H}t\right)\hat{x}\exp\left(-\imath\hat{H}t\right)$ with $\hat{x}$ being at time $t = 0$. Now, if $\left|\psi_i\right\rangle$ and $\left|\psi_f\right\rangle$ are two arbitrary states with nonzero projections on $|0\rangle$, then a ratio of the type $\langle 0|\,\hat{O}_1\,|0\rangle\,/\,\langle 0|\,\hat{O}_2\,|0\rangle$, where $\hat{O}_1$ and $\hat{O}_2$ are generic operators, turns out to be equal to

$$\lim_{T_i,T_f\to\infty,\varepsilon\to 0}\frac{\left\langle\psi_f\right|e^{-\imath T_f\hat{H}(1-\imath\varepsilon)}\,\hat{O}_1e^{-\imath T_i\hat{H}(1-\imath\varepsilon)}\left|\psi_i\right\rangle}{\left\langle\psi_f\right|e^{-\imath T_f\hat{H}(1-\imath\varepsilon)}\,\hat{O}_2e^{-\imath T_i\hat{H}(1-\imath\varepsilon)}\left|\psi_i\right\rangle},$$

for if $|n\rangle$ is an energy eigenstate with energy E_n, then

$$\begin{aligned} e^{-\iota T_i \hat{H}(1-\iota\varepsilon)} \left|\psi_i\right\rangle &= \sum_n e^{-\iota T_i \hat{H}(1-\iota\varepsilon)} \left|n\right\rangle \left\langle n \middle| \psi_i \right\rangle \\ &= \sum_n e^{-\iota T_i E_n (1-\iota\varepsilon)} \left|n\right\rangle \left\langle n \middle| \psi_i \right\rangle \\ &\to\ e^{-\iota T_i E_0 (1-\iota\varepsilon)} \left|0\right\rangle \left\langle 0 \middle| \psi_i \right\rangle \end{aligned}$$

if $\varepsilon \to 0$ and $T_i \to \infty$. Of course, we assume here that $E_0 = 0$ (we shift $\hat{H}$ by an appropriate constant if such is not the case), the ground state is nondegenerate, and there is an energy gap between it and the first excited state. Then, setting $\hat{O}_1$ equal to

$$\hat{x} e^{\iota \hat{H}(t_2 - t_1)} \hat{x} e^{\iota \hat{H}(t_3 - t_2)} \ldots \hat{x}$$

and $\hat{O}_2$ equal to $e^{-\iota \hat{H}(t_1 - t_n)(1-\iota\varepsilon)}$, we obtain

$$\lim_{T_i, T_f \to \infty, \varepsilon \to 0} \frac{\left\langle \psi_f \right| e^{-\iota T_f \hat{H}(1-\iota\varepsilon)} \hat{x} e^{\iota \hat{H}(t_2 - t_1)} \hat{x} e^{\iota \hat{H}(t_3 - t_2)} \ldots \hat{x} e^{-\iota T_i \hat{H}(1-\iota\varepsilon)} \left|\psi_i\right\rangle}{\left\langle \psi_f \right| e^{-\iota \hat{H}(T_f + T_1 + t_1 - t_n)(1-\iota\varepsilon)} \left|\psi_i\right\rangle}.$$

If we insert completeness relations at appropriate places now, replace the time-evolution operators by path integrals, and do the T_i and T_f limits, we reach

$$\langle \hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n)\rangle = \lim_{\varepsilon \to 0} \frac{\int_{x_i}^{x_f} [dx(t)] \psi_f^*(x_f) \psi_i(x_i) x(t_1) \ldots x(t_n) \exp(\iota S_\varepsilon[x(t)])}{\int_{x_i}^{x_f} [dx(t)] \psi_f^*(x_f) \psi_i(x_i) \exp(\iota S_\varepsilon[x(t)])},$$

where S_ε denotes the action obtained by replacing t with $t(1 - \iota\varepsilon)$ and x_i and x_f are taken at $t \to \mp\infty$, respectively. Since the wavefunctions ψ_i and ψ_f are arbitrary, we may set both $\psi_i(x_i)$ and

$\psi_f(x_f)$ equal to 1, thereby getting

$$\langle \hat{x}(t_1)\hat{x}(t_2)\ldots\hat{x}(t_n)\rangle = \lim_{\varepsilon\to 0}\frac{\int[dx(t)]x(t_1)\ldots x(t_n)\exp(\iota S_\varepsilon[x(t)])}{\int[dx(t)]\exp(\iota S_\varepsilon[x(t)])}. \tag{2.2}$$

It might seem that evaluating correlators this way is as cumbersome as—if not more than—evaluating them with Hamiltonian field theory. However, we show why such is not the case. The expression in Eq. (2.2) could be adapted to the case of fields by simply replacing x with φ everywhere, and the result obtained thereupon, that is,

$$\langle \hat{\varphi}(t_1)\hat{\varphi}(t_2)\ldots\hat{\varphi}(t_n)\rangle = \lim_{\varepsilon\to 0}\frac{\int[d\varphi(t)]\varphi(t_1)\ldots\varphi(t_n)\exp(\iota S_\varepsilon[\varphi(t)])}{\int[d\varphi(t)]\exp(\iota S_\varepsilon[\varphi(t)])},$$

could then be *generated* by something known as a *generating functional* via the technique of functional differentiation. This very technique reduces calculating any correlator to merely taking derivatives.

A generating functional could be understood as being a path integral with a parameter (also known as auxiliary "current") that one could repeatedly differentiate with respect to in order to get progressively higher correlators. For example, the path integral

$$Z[J] = \int[d\varphi]\exp\left(\iota S_\varepsilon[\varphi] + \iota\int d^4x J(x)\varphi(x)\right) \tag{2.3}$$

could serve as the generating functional for the case at hand. $J(x)$ here is the aforementioned parameter. Differentiating with respect to it once is like

$$\frac{\partial}{\partial J(x_1)}Z[J] = \iota\int[d\varphi]\varphi(x_1)\exp\left(\iota S_\varepsilon[\varphi] + \iota\int d^4x J(x)\varphi(x)\right),$$

twice is like

$$\frac{\delta}{\delta J(x_1)}\frac{\delta}{\delta J(x_2)}Z[J] = \iota^2 \int [d\varphi]\varphi(x_1)\varphi(x_2)\exp\left(\iota S_\varepsilon[\varphi] + \iota\int d^4x J(x)\varphi(x)\right),$$

and n times is like

$$\frac{\delta}{\delta J(x_1)}\cdots\frac{\delta}{\delta J(x_n)}Z[J] = \iota^n \int [d\varphi]\varphi(x_1)\ldots\varphi(x_n)\exp\left(\iota S_\varepsilon[\varphi] + \iota\int d^4x J(x)\varphi(x)\right).$$

The pattern emerging with these successive derivatives allows us to see that

$$\langle\hat{\varphi}(x_1)\hat{\varphi}(x_2)\ldots\hat{\varphi}(x_n)\rangle = Z[0]^{-1}\frac{1}{\iota}\frac{\delta}{\delta J(x_1)}\cdots\frac{1}{\iota}\frac{\delta}{\delta J(x_n)}Z[J]\bigg|_{J=0}. \tag{2.4}$$

The implications of this result happen to be quite remarkable, for what it essentially means is that given the action $S[\varphi]$, one needs to evaluate the generating functional in Eq. (2.3) only, and then, any correlator could be worked out using Eq. (2.4).

2.2 Free bosonic field

As stated in section 1.3, the Klein-Gordon (KG) field is the simplest to be studied both classically and quantum mechanically, and it also serves as the prime example of a free bosonic field. It is free in that its Lagrangian density,

$$\mathcal{L} = \frac{1}{2}(\partial_\mu\varphi)^2 - \frac{1}{2}m^2\varphi^2,$$

does not have terms with powers of φ greater than 2 or terms involving fields beside φ; it is these terms that correspond to interacting particles and, thus, interacting fields. The bosonic nature of the field comes from the commutation properties of the creation/annihilation operators used in its mode expansions, and to calculate its correlators via the Hamiltonian field theory, we refer back to these very mode expansions, which we introduced in section 1.3, but write them in the Heisenberg—rather than the Schrodinger—picture this time:

$$\hat{\varphi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \left(\hat{a}_\mathbf{p} e^{-\imath px} + \hat{a}^\dagger_\mathbf{p} e^{\imath px} \right)\Big|_{p^0=\omega_\mathbf{p}}$$

and

$$\hat{\pi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} (-\imath) \sqrt{\frac{\omega_\mathbf{p}}{2}} \left(\hat{a}_\mathbf{p} e^{-\imath px} - \hat{a}^\dagger_\mathbf{p} e^{\imath px} \right)\Big|_{p^0=\omega_\mathbf{p}} .$$

Then, we could write a 2-point correlator for the KG field as

$$\begin{aligned} \langle \hat{\varphi}(x)\hat{\varphi}(y) \rangle &= \langle 0 | \mathcal{T}(\hat{\varphi}(x)\hat{\varphi}(y)) | 0 \rangle \\ &= \theta(x_0 - y_0) D(x-y) + \theta(y_0 - x_0) D(y - x), \end{aligned} \tag{2.5}$$

where θ stands for the usual Heaviside function and helps put the action of the time ordering operator into mathematical form. $D(x-y)$, on the other hand, stands for $\langle 0 | \hat{\varphi}(x)\hat{\varphi}(y) | 0 \rangle$, which, when worked out using the mode expansions above, turns out to be equal to

$$\int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{2E_\mathbf{p}} e^{-\imath p(x-y)}\Big|_{p^0=E_\mathbf{p}} .$$

Here, we have replaced $\omega_\mathbf{p}$ by $E_\mathbf{p}$ since $E_\mathbf{p} = \hbar\omega_\mathbf{p}$ and we set $\hbar$ equal to 1 everywhere. Now, if $x_0 > y_0$, then the correlator required is $D(x-y)$, and if $x_0 < y_0$, the correlator is $D(y-x)$. However, this thing amounts to having two different expressions for same 2-point correlator. Working to find a single expression, we realize that

$$D_R(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{\imath}{p^2 - m^2} e^{-\imath p(x-y)}$$

could be used with the pole prescription in Fig. 2.1 to evaluate the correlator in both of the cases above. When $x_0 > y_0$, we perform the p^0 integral by closing the contour below, and we obtain $D(x-y)$; when $x_0 < y_0$, we do the p^0 integral by closing the contour above, thereby obtaining $D(y-x)$.

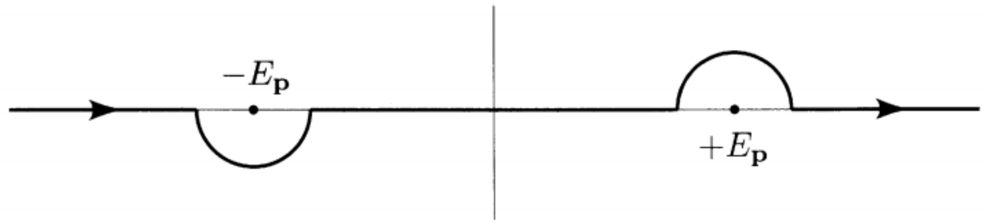


Figure 2.1: Pole prescription for the KG-field 2-point correlator. When $x_0 > y_0$, the contour is closed below; when $x_0 < y_0$, the contour is closed above.

A convenient way to remember this prescription, also known as the *Feynman prescription*, is to write

$$D_F(x-y) = \int \frac{d^4p}{(2\pi)^4} \frac{\imath}{p^2 - m^2 + \imath\varepsilon} e^{-\imath p(x-y)}, \tag{2.6}$$

for the poles are then given by $p^0 = \pm(E_{\mathbf{p}} - \imath\varepsilon)$, displaced appropriately below and above the real axis. Thus, we could finally say that $\langle\hat{\varphi}(x)\hat{\varphi}(y)\rangle = D_F(x-y)$, something which amounts to the fact that we have successfully calculated the 2-point correlator for the KG field (or a free bosonic field) via the Hamiltonian field theory.

Proceeding to work it out now via the path integral formalism, we begin by introducing the following Fourier and inverse Fourier transforms:

$$\tilde{\varphi}(k) = \int d^4x e^{\imath kx} \varphi(x)$$

and

$$\varphi(x) = \int \frac{d^4k}{(2\pi)^4} e^{-\imath kx} \tilde{\varphi}(k).$$

Then, using them in $S_0 = S + \int d^4 x J\varphi$, with S being

$$\int d^4 x \mathcal{L} = \int d^4 x \left[\frac{1}{2}(\partial_\mu \varphi)^2 - \frac{1}{2}m^2\varphi^2\right],$$

we obtain

$$S_0 = \frac{1}{2}\int \frac{d^4 k}{(2\pi)^4}\left[\tilde{\varphi}(k)(k^2 - m^2)\tilde{\varphi}(-k) + \tilde{J}(k)\tilde{\varphi}(-k) + \tilde{J}(-k)\tilde{\varphi}(k)\right].$$

In these calculations, J is the what we referred to as an auxiliary current in section 2.1, and we have Fourier transformed it just as we Fourier transformed φ. If we now change variables via

$$\tilde{\chi}(k) = \tilde{\varphi}(k) + \frac{\tilde{J}(k)}{k^2 - m^2},$$

we could rewrite S_0 as

$$S_0 = \frac{1}{2}\int \frac{d^4 k}{(2\pi)^4}\left[-\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2} + \tilde{\chi}(k)(k^2 - m^2)\tilde{\chi}(-k)\right].$$

Since we want to find the 2-point correlator by way of path integrals this time, we need, as we established in section 2.1, a generating functional, and we use the one we gave in Eq. (2.3). It should be noted that the argument of the exponential—barring the $\imath$—is the same as S_0, and we thus use the expression we found for S_0 to get

$$\begin{aligned}
Z[J] &= \int [d\chi] \exp\left\{\frac{\imath}{2}\int \frac{d^4 k}{(2\pi)^4}\left[-\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2} + \tilde{\chi}(k)(k^2 - m^2)\tilde{\chi}(-k)\right]\right\} \\
&= \exp\left(-\frac{\imath}{2}\int \frac{d^4 k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2}\right)\exp\left(\frac{\imath}{2}\int [d\chi]\int \frac{d^4 k}{(2\pi)^4}\tilde{\chi}(k)(k^2 - m^2)\tilde{\chi}(-k)\right) \\
&= Z[0]\exp\left(-\frac{\imath}{2}\int \frac{d^4 k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2 - m^2}\right),
\end{aligned}$$

where we used the change of variables from φ to χ in the first line and the fact that the exponential without any J is simply $Z[J]$ with J set equal to 0 in the third line. At this point, we broach that we used $t(1-\iota\varepsilon)$ instead of t in section 2.1; doing so has the same effect as replacing m^2 with $m^2-\iota\varepsilon$ everywhere, and hence, we could write our generating functional as

$$Z[J] = Z[0]\exp\left(-\frac{\iota}{2}\int\frac{d^4k}{(2\pi)^4}\frac{\tilde{J}(k)\tilde{J}(-k)}{k^2-m^2+\iota\varepsilon}\right),$$

which could be further simplified to get to

$$Z[J] = Z[0]\exp\left(\frac{\iota}{2}\int d^4x d^4x' J(x)\Delta(x-x')J(x')\right). \tag{2.7}$$

Here, we have used the definition

$$\Delta(x-x') = \int\frac{d^4k}{(2\pi)^4}\frac{-e^{-\iota k(x-x')}}{k^2-m^2+\iota\varepsilon}$$

and put our generating functional into a rather suggestive form, one which gives the hint that we are right on the track to finding correlators via path integrals. Then, using Eq. (2.4), we have

$$\begin{aligned}\langle\hat{\varphi}(x_1)\hat{\varphi}(x_2)\rangle &= Z[0]^{-1}\frac{1}{\iota}\frac{\partial}{\partial J(x_1)}\frac{1}{\iota}\frac{\partial}{\partial J(x_2)}Z[J]\bigg|_{J=0} \\ &= \frac{1}{\iota}\frac{\partial}{\partial J(x_1)}\left[\left(\int d^4x'\Delta(x'-x_2)J(x')\right)Z[J]\right]\bigg|_{J=0} \\ &= \frac{1}{\iota}\Delta(x_1-x_2) \\ &= \int\frac{d^4k}{(2\pi)^4}\frac{\iota}{k^2-m^2+\iota\varepsilon}e^{-\iota k(x_1-x_2)},\end{aligned} \tag{2.8}$$

which is the same as $D_F(x_1-x_2)$ except for its having k in place of p. However, this thing is not an issue since $p=\hbar k$ and we set $\hbar$ equal to 1 everywhere. Clearly then, we have been able to calculate the 2-point correlator for the KG field both by Hamiltonian field theory and by path integral formalism. Particularly important to note is the ease that the latter provides us with; for instance, we could simply continue differentiation as shown in Eq. (2.8) to obtain higher and higher correlators.

2.3 Free fermionic field

Just as the KG field serves as the archetypal example of a free bosonic field, so does the Dirac field, as discussed in section 1.3, form the typical example given when it comes to free fermionic fields. Governed by the Lagrangian density

$$\mathcal{L} = \bar{\psi}(\imath\gamma^\mu \partial_\mu - m)\psi,$$

the Dirac field is free in that its Lagrangian density is bilinear in ψ and $\bar{\psi}$ and does not have terms with fields beside either of the two. The fermionic nature of the field is encapsulated by the anti-commutation properties of the creation/annihilation operators we introduce in its mode expansions, which we go back to in order to use the Hamiltonian field theory to calculate the correlators. As presented before (section 1.3), the mode expansions are given by

$$\hat{\psi}(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \sum_{s=1}^{2} \left(\hat{a}^s_\mathbf{p} u^s(\mathbf{p}) e^{-\imath px} + \hat{b}^{s\dagger}_\mathbf{p} v^s(\mathbf{p}) e^{\imath px} \right)\Big|_{p^0=\omega_\mathbf{p}}$$

and

$$\hat{\psi}^\dagger(x) = \int \frac{d^3\mathbf{p}}{(2\pi)^3} \frac{1}{\sqrt{2\omega_\mathbf{p}}} \sum_{s=1}^{2} \left(\hat{b}^s_\mathbf{p} v^{s\dagger}(\mathbf{p}) e^{\imath px} + \hat{a}^{s\dagger}_\mathbf{p} u^{s\dagger}(\mathbf{p}) e^{-\imath px} \right)\Big|_{p^0=\omega_\mathbf{p}}.$$

The only difference is that we have chosen to write them in the Heisenberg picture, instead of the Schrodinger picture, this time around.

The 2-point correlator for the field could be defined as

$$\left\langle \hat{\psi}(x)\bar{\hat{\psi}}(y) \right\rangle = \langle 0| \mathcal{T}\left(\hat{\psi}(x)\bar{\hat{\psi}}(y)\right)|0\rangle$$
$$= \theta(x_0 - y_0)S(x-y) - \theta(y_0 - x_0)\bar{S}(y-x), \tag{2.9}$$

where θ is the Heaviside function, $S(x-y)$ is $\langle 0| \hat{\psi}(x)\bar{\hat{\psi}}(y)|0\rangle$, and $\bar{S}(y-x)$ is $\langle 0| \bar{\hat{\psi}}(y)\hat{\psi}(x)|0\rangle$. When worked out using the mode expansions above, $\langle 0| \hat{\psi}(x)\bar{\hat{\psi}}(y)|0\rangle$ and $\langle 0| \bar{\hat{\psi}}(y)\hat{\psi}(x)|0\rangle$ yield

$$\langle 0| \hat{\psi}(x)\bar{\hat{\psi}}(y)|0\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\,(\gamma^\mu p_\mu + m)e^{-\iota p(x-y)}\Big|_{p^0=E_\mathbf{p}}$$

and

$$\langle 0| \bar{\hat{\psi}}(y)\hat{\psi}(x)|0\rangle = \int \frac{d^3\mathbf{p}}{(2\pi)^3}\frac{1}{2E_\mathbf{p}}\,(\gamma^\mu p_\mu - m)e^{\iota p(x-y)}\Big|_{p^0=E_\mathbf{p}},$$

respectively, where we have replaced $\omega_\mathbf{p}$ with $E_\mathbf{p}$ everywhere for the same reason as in the last section and used $\sum_s u^s(\mathbf{p})\bar{u}^s(\mathbf{p}) = \gamma^\mu p_\mu + m$ and $\sum_s v^s(\mathbf{p})\bar{v}^s(\mathbf{p}) = \gamma^\mu p_\mu - m$. Once again, if $x_0 > y_0$, then the correlator is $S(x-y)$, and if $x_0 < y_0$, the correlator is $\bar{S}(y-x)$. What we need, however, is a single expression for the 2-point correlator at hand, and we note that

$$S_R(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{\iota}{p^2 - m^2}(\gamma^\mu p_\mu + m)e^{-\iota p(x-y)}$$

could be simplified with the pole prescription in Fig. 2.1 to get the correlator in either of the cases. When $x_0 > y_0$, the p^0 integral should be done with the contour closed below to obtain $S(x-y)$; when $x_0 < y_0$, the p^0 integral should be done with the contour closed above to obtain $\bar{S}(y-x)$. Much more convenient, as was the case for the KG field too, is to use the Feynman prescription and write

$$S_F(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{\iota}{p^2 - m^2 + \iota\varepsilon}(\gamma^\mu p_\mu + m)e^{-\iota p(x-y)}$$

since the poles are then $p^0 = \pm(E_{\mathbf{p}} - \iota\varepsilon)$, that is, they are displaced appropriately below and above the real axis as shown in Fig. 2.2.

Figure 2.2: Feynman prescription for 2-point correlators. When $x_0 > y_0$, contour C_F is closed below; when $x_0 < y_0$, contour C_F is closed above.

Hence, we could finally write $\left\langle \hat{\psi}(x)\bar{\hat{\psi}}(y) \right\rangle = S_F(x-y)$, which means that we have successfully evaluated the 2-point correlator for the Dirac field (or a free fermionic field) via the Hamiltonian field theory and can now proceed to working it out via the path integral formalism.

However, before subjecting fermionic fields to the machinery of functional differentiation, we need to ensure that we have something incorporating their anticommutation properties, and we get this thing from the so-called *Grassmann algebra*. It could be defined as a vector space, $\mathcal{V}$, constructed from, say, n generators θ_i that satisfy the antisymmetry product

$$\theta_i\theta_j + \theta_j\theta_i = 0. \tag{2.10}$$

A generic element of a Grassmann algebra is hence a first-degree polynomial in the generators θ_i, that is,

$$f(\theta_i) = \sum_{k=0}^{n} \sum_{i_1,\ldots,i_k}^{n} C^{(k)}_{i_1,\ldots,i_k} \theta_{i_1} \ldots \theta_{i_k},$$

where the coefficients $C^{(k)}_{i_1,\ldots,i_k}$ are defined only if all their indices happen to be different and a standard ordering is defined on these indices. What these definitions imply is that the dimension of a Grassmann algebra equals the number of distinct monomials that could be constructed from the θ_i, namely 2^n. Just as examples, we could consider the generic elements of Grassmann algebras with

$n = 1$ and $n = 2$.

$$n = 1 \rightarrow f(\theta) = c_0 + c_1\theta;$$

$$n = 2 \rightarrow f(\theta_1, \theta_2) = c_0 + c_1\theta_1 + c_2\theta_2 + c_{12}\theta_1\theta_2.$$

Clearly, any other terms added to these expressions would either be redundant or be zero as per the anticommutation properties of the θ_i.

The θ_i are often called Grassmann variables, and the elements of the algebra, being polynomials in the θ_i, are often referred to as functions of Grassmann variables. With functions in hand, we could proceed to defining differentiation on a Grassmann algebra by treating the θ_i as ordinary variables with anticommutation properties appended. A convention must therefore be adopted: the variable a function is being differentiated with respect to must be brought to the left of every expression before the derivative is taken. As an example, we could consider the derivatives of $f(\theta_1, \theta_2)$, a function of Grassmann variables which we gave above.

$$\frac{\partial}{\partial\theta_1}(f(\theta_1, \theta_2)) = c_1 + c_{12}\theta_2;$$
$$\frac{\partial}{\partial\theta_2}(f(\theta_1, \theta_2)) = c_2 - c_{12}\theta_1.$$

Since functions of Grassmann numbers are at most linear in the θ_i, the differential operator $\partial/\partial\theta_i$ is nilpotent, that is, $(\partial/\partial\theta_i)^2 = 0$. In fact, these operators, together with the θ_i, form what is known as a *Clifford algbera*, that is, the following are satisfied.

$$\begin{aligned} \theta_i\theta_j + \theta_j\theta_i &= 0; \\ \frac{\partial}{\partial\theta_i}\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\frac{\partial}{\partial\theta_i} &= 0; \\ \theta_i\frac{\partial}{\partial\theta_j} + \frac{\partial}{\partial\theta_j}\theta_i &= \delta_{ij}. \end{aligned} \tag{2.11}$$

We end our discussion of Grassmann numbers here, for we have established everything we need for our primary task: calculating a 2-point correlator for the Dirac field. This time, we take a somewhat roundabout route that does not precisely use any path integrals with fermionic fields but takes us to the essential thing that a path integral would lead to, that is, a generating functional. This functional could then be used with functional differentiation to generate correlators as desired. We go this indirect way to show that a generating functional for correlators need not always be worked out directly from the action.

We start with the functional

$$Z[\eta, \bar{\eta}] = \langle 0| \mathcal{T} \exp\left[\int d^4x \left(\bar{\eta}(x)\hat{\psi}(x) + \bar{\hat{\psi}}(x)\eta(x)\right)\right] |0\rangle ,$$

where η and $\bar{\eta}$ are spinors (just like ψ and $\bar{\psi}$) that play the same role as that played by an auxiliary current. An important thing to note is that the functions η and $\bar{\eta}$ are Grassmann-valued, and we take them as anticommuting with ψ and $\bar{\psi}$ as well. We would also assume that functional derivative operators involving η and $\bar{\eta}$ comprise a Clifford algebra with η, $\bar{\eta}$, ψ, and $\bar{\psi}$. It is easy to see how functional differentiation could be used to generate the correlator $\left\langle \hat{\psi}(x_1)\bar{\hat{\psi}}(x_2)\right\rangle$ in the following way:

$$\left\langle \hat{\psi}(x_1)\bar{\hat{\psi}}(x_2)\right\rangle = Z[0,0]^{-1}\frac{\delta}{\delta\bar{\eta}(x_2)}\frac{\delta}{\delta\eta(x_1)}Z[\eta,\bar{\eta}]\bigg|_{\eta=0,\bar{\eta}=0} .$$

If we now begin with

$$\frac{\delta Z}{\delta\bar{\eta}(x)} = \langle 0| \mathcal{T}\left(\hat{\psi}(x)e^{\int d^4y\left(\bar{\eta}(y)\hat{\psi}(y)+\bar{\hat{\psi}}(y)\eta(y)\right)}\right)|0\rangle$$

and apply $\imath\gamma^\mu\partial_\mu - m$ to both sides, we end up with

$$(\imath\gamma^\mu\partial_\mu - m)\frac{\delta Z}{\delta\bar{\eta}(x)} = \imath\eta(x)Z,$$

which could be shown to be satisfied by

$$Z[\eta,\bar{\eta}] = Z[0,0]\exp\left(\int d^4x d^4y \bar{\eta}(x) S_F(x-y)\eta(y)\right) \tag{2.12}$$

since $(\iota\gamma^\mu\partial_\mu - m)S_F(x-y) = \iota\delta^{(4)}(x-y)$. Then,

$$\begin{aligned}
\langle 0|\,\mathcal{T}\left(\hat{\psi}(x_1)\bar{\hat{\psi}}(x_2)\right)|0\rangle &= Z[0,0]^{-1}\,\frac{\partial}{\partial\bar{\eta}(x_2)}\frac{\partial}{\partial\eta(x_1)}Z[\eta,\bar{\eta}]\Big|_{\eta=0,\bar{\eta}=0} \\
&= -Z[0,0]^{-1}\,\frac{\partial}{\partial\bar{\eta}(x_2)}\left(\int d^4x'\bar{\eta}(x')S_F(x'-x_1)\right)Z[\eta,\bar{\eta}]\Big|_{\eta=0,\bar{\eta}=0} \\
&= S_F(x_1-x_2) \\
&= \int\frac{d^4p}{(2\pi)^4}\frac{\iota}{p^2-m^2+\iota\varepsilon}(\gamma^\mu p_\mu + m)e^{-\iota p(x_1-x_2)}.
\end{aligned} \tag{2.13}$$

Seeing that this result is precisely the same as the one obtained from the Hamiltonian field theory, we could say that we have been successful in finding the 2-point correlator for the Dirac field via the path integral formalism too. Again, we notice the ease that path integrals afford: simply continuing to differentiate Eq. (2.12) allows us to find progressively higher correlators.

2.4 Wick's theorem

This section is devoted to simply further demonstrating the facility that the technique of functional differentiation accouters us with. It does so by showing how naturally the famous *Wick's theorem* could be seen in terms of functional derivatives.

What Wick's theorem actually does is that it connects the operation of time ordering with the operation of *normal ordering*, which is represented by the so-called *normal ordering* operator $\mathcal{N}$. This operator arranges the factors following it (which themselves happen to be operators and field operators to be precise) in such a way that all the creation operators are brought to the left of all the annihilation operators. The way Wick's theorem connects $\mathcal{T}$ and $\mathcal{N}$ could best be shown by an example. For the field φ, let us define φ_1 to be equal to $\varphi(x_1)$, φ_2 to be equal to $\varphi(x_2)$, and so on up to φ_n. According to Wick's theorem then,

$$\mathcal{T}(\varphi_1\varphi_2\dots\varphi_n) = \mathcal{N}(\varphi_1\varphi_2\dots\varphi_n) + \mathcal{N}(\text{All possible terms with Wick contractions}), \tag{2.14}$$

where a *Wick contraction* refers to the replacement of a pair of fields with the corresponding 2-point correlator. To nail this point even more, we consider

$$\begin{aligned}\mathcal{T}(\varphi_1\varphi_2\varphi_3\varphi_4) = {} & \mathcal{N}(\varphi_1\varphi_2\varphi_3\varphi_4) + \mathcal{N}(\wick{\c\varphi_1\c\varphi_2}\varphi_3\varphi_4) + \mathcal{N}(\wick{\c\varphi_1\varphi_2\c\varphi_3}\varphi_4) + \mathcal{N}(\wick{\c\varphi_1\varphi_2\varphi_3\c\varphi_4}) \\ & + \mathcal{N}(\varphi_1\wick{\c\varphi_2\c\varphi_3}\varphi_4) + \mathcal{N}(\varphi_1\wick{\c\varphi_2\varphi_3\c\varphi_4}) + \mathcal{N}(\varphi_1\varphi_2\wick{\c\varphi_3\c\varphi_4}) + \wick{\c1\varphi_1\c1\varphi_2\c2\varphi_3\c2\varphi_4} \\ & + \wick{\c1\varphi_1\c2\varphi_2\c1\varphi_3\c2\varphi_4} + \wick{\c1\varphi_1\c2\varphi_2\c2\varphi_3\c1\varphi_4}.\end{aligned}$$

Each of the lines connecting the fields represents a Wick contraction, meaning that the fields involved would be replaced by their correlator. With the description of Wick's theorem done, we now go to show how we make functional derivatives enter the picture. Beginning by considering

$$\exp\left(\int \frac{1}{2}G\partial\partial\right)(\varphi(1)\dots\varphi(N)),$$

where $\varphi(i)$ stands for $\varphi(x_i)$ and

$$\int \frac{1}{2}G\partial\partial$$

stands for

$$\int_{x,y} \frac{1}{2}G(x,y)\frac{\partial}{\partial\varphi(x)}\frac{\partial}{\partial\varphi(y)}$$

with $G(x,y)$ representing a general 2-point correlator for the field φ, we could do the following calculation:

$$\int \frac{1}{2} G\partial\partial\varphi(1)\varphi(2) = \frac{1}{2}\bigg(\int_x G(x,1)\frac{\partial}{\partial\varphi(x)}\varphi(2) + \int_x G(x,2)\frac{\partial}{\partial\varphi(x)}\varphi(1)\bigg) = G(1,2).$$

It clearly shows that application of $\int(1/2)G\partial\partial$ replaces the two fields with their correlator. Making the calculation a little general then, we see that

$$\int \frac{1}{2} G\partial\partial\varphi(1)\varphi(2)\ldots\varphi(N) = \sum_{i<j} G(i,j)\varphi(1)\ldots\varphi(i-1)\varphi(i+1)\ldots \varphi(j-1)\varphi(j+1)\ldots\varphi(N).$$

As is evident, applying a single power of $\int(1/2)G\partial\partial$ to a collection of fields gives all possible forms of that collection with a single Wick contraction. This thing motivates us to see the application of the second term in the expansion of $\exp\left[\int(1/2)G\partial\partial\right]$ to a particular set of fields:

$$\frac{1}{2!}\int \frac{1}{2} G\partial\partial \int \frac{1}{2} G\partial\partial\varphi(1)\varphi(2)\varphi(3)\varphi(4) = G(1,2)G(3,4) + G(1,3)G(2,4) + G(1,4)G(2,3).$$

Clearly, applying two powers of $\int(1/2)G\partial\partial$ as they appear in the expansion of $\exp\left[\int(1/2)G\partial\partial\right]$ to a collection of fields yields all possible forms of that collection with two Wick contractions. In fact, comparing the result just obtained with the concrete example we gave for Wick's theorem shows that we have all possible terms with two Wick contractions.

Continuing this very way, we obtain the result

$$\begin{aligned}\exp\left(\int \frac{1}{2} G\partial\partial\right)\varphi(1)\ldots\varphi(N) = \varphi(1)\ldots\varphi(N) \\ + \sum \text{Terms with 1 Wick contraction} \\ + \sum \text{Terms with 2 Wick contractions} \\ + \ldots, \end{aligned} \tag{2.15}$$

which essentially says that in order to apply Wick's theorem to a time ordered set of fields, subject that set to an application by $\exp\left[\int (1/2)G\partial\partial\right]$ and normal order the result. What is of particular interest here is that one gets all possible contractions automatically and does not have to think as to which ones are left. We may, in fact, refer to this description of Wick's theorem as a particularly neat derivation of it. It is the technique of functional differentiation that does the trick here, and it is this very technique that makes path integrals so useful when it comes to using them to calculate correlators.

This chapter was primarily devoted to the calculation of free bosonic and fermionic correlators via both Hamiltonian field theory and path integral formalism. It is true that working correlators out is important from the standpoint of physical interest, but we had another reason, which we have not brought up as yet, to study them in detail. As of next chapter, the work enters the discussion of conformal field theories, which are quantum field theories that correlators play a particularly important role in, so much so that it is possible in some cases to completely define such theories solely on the basis of symmetry properties of the correlators and without reference to any Lagrangians or actions. With this slight motivation for what is up next, we end this chapter on correlators, looking forward to the discussion on the subset of quantum field theories called conformal field theories.

"Symmetry is what we see at a glance."

Blaise Pascal

3

Conformal field theory[1,2]

Beginning with a general discussion of the meaning of a quantum field (chapter 0), this work moved to detailing some quantization techniques (chapter 1), which essentially are procedures taking one from the world of classical fields to the realm of quantum ones. Thereafter, it proceeded to using quantum fields to calculate physically important quantities known as correlators (chapter 2), and the calculations were done for both bosonic and fermionic fields. As from now, the work shifts from studying quantum field theories in general to poring over a particular subset of them that quantities like correlators form an extremely important part of—so much so that it is sometimes possible to completely define and solve such theories through the symmetry properties of their correlators only. This subset consists of theories called conformal field theories, or CFTs for short, and their defining feature is that they show some sort of invariance under what are known as conformal transformations.

The first goal of the current chapter is to discuss these very transformations and the kind of symmetry group they form. It then presents these transformations in the context of both classical and quantum fields. Finally, it includes a detailed consideration of 2-dimensional CFTs, which are particularly interesting due to the many constraints they come with.

3.1 The conformal group

Denoting by $g_{\mu\nu}$ the metric of a spacetime of dimension d, we could define a *conformal transformation* of the coordinates as an invertible mapping, $x \to x'$, that leaves the metric invariant up to some positive scale Λ:

$$g'_{\mu\nu}(x') = \Lambda(x) g_{\mu\nu}(x). \tag{3.1}$$

Put in some simpler terms, a conformal transformation is equivalent at every point of spacetime to a rotation and a dilation. In an even more simplified manner, conformal transformations could be called the transformations that preserve angles. Manifestly forming a group, the set of conformal transformations definitely has the Poincare group as a subgroup, for this latter group corresponds to the special case $\Lambda = 1$.

We begin our study of conformal transformations by working out the constraints that Eq. (3.1) puts on an infinitesimal transformation of coordinates, $x^\mu \to x'^\mu = x^\mu + \varepsilon^\mu(x)$. To first order, the change in the metric is given by

$$g'_{\mu\nu} = g_{\mu\nu} - (\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu),$$

which, when seen jointly with Eq. (3.1), requires that

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = f(x) g_{\mu\nu}.$$

$f(x)$ here could be found in terms of the $\varepsilon^\mu(x)$ by tracing both sides, a fact which allows us to rewrite the equation above as

$$\partial_\mu \varepsilon_\nu + \partial_\nu \varepsilon_\mu = \frac{2}{d}(\partial_\rho \varepsilon^\rho) g_{\mu\nu}. \tag{3.2}$$

This relation does comprise constraint for the infinitesimal $\varepsilon^\mu(x)$, but to determine the explicit form of conformal transformations in d dimensions, we need some further constraints as well, one of which could be derived from subjecting Eq. (3.2) to application by ∂_ρ, permuting indices to get three equations, and taking a certain linear combination (subtracting one of the equations from the sum of the other two):

$$2\partial_\mu \partial_\nu \varepsilon_\rho = \frac{2}{d}(g_{\mu\rho}\partial_\nu + g_{\nu\rho}\partial_\mu - g_{\mu\nu}\partial_\rho)(\partial_\lambda \varepsilon^\lambda). \tag{3.3}$$

Another constraint could be obtained by contracting this one with $g^{\mu\nu}$ and applying $\Box = \partial_\mu \partial^\mu$ to both sides:

$$(2-d)\partial_\mu \partial_\nu (\partial_\rho \varepsilon^\rho) = g_{\mu\nu} \Box (\partial_\rho \varepsilon^\rho).$$

It could be written in a more compact form, which is the form that we actually use, by contracting with $g^{\mu\nu}$ once again:

$$(d-1)\Box(\partial_\rho \varepsilon^\rho) = 0. \tag{3.4}$$

Eqs. (3.2), (3.3), and (3.4) thus put constraints on the infinitesimal $\varepsilon^\mu(x)$, and we could see what those constraints are for different possible d. First, when $d = 1$, there are no constraints on $\varepsilon^\mu(x)$, meaning that any smooth transformation in 1 dimension is conformal. We leave the case $d = 2$ for a later discussion. The case $d \geq 3$ then becomes our focus for now.

When $d \geq 3$, Eq. (3.4) implies that $\Box(\partial_\rho \varepsilon^\rho) = 0$. Since $\Box = \partial_\mu \partial^\mu$, we could say that $\partial_\rho \varepsilon^\rho$ is at most linear in the coordinates. This thing, in turn, tells us about ε^μ's being at most quadratic in the coordinates, thereby allowing us to write

$$\varepsilon_\mu = a_\mu + b_{\mu\nu} x^\nu + c_{\mu\nu\rho} x^\nu x^\rho, \tag{3.5}$$

with $c_{\mu\nu\rho} = c_{\mu\rho\nu}$ and $|a_\mu|, |b_{\mu\nu}|, |c_{\mu\nu\rho}| \ll 1$. Now, since the constraint equations hold for all space-time points, we may consider each power of the coordinates independently. The constant term a^μ is not constrained by any of the constraint equations, and it represents an infinitesimal translation. When the term linear in x^μ, that is, $b_{\mu\nu} x^\nu$, is put in Eq. (3.2), we obtain

$$b_{\mu\nu} + b_{\nu\mu} = \frac{2}{d} b^\lambda{}_\lambda n_{\mu\nu},$$

which implies that $b_{\mu\nu}$ is the sum of an antisymmetric part and a pure trace:

$$b_{\mu\nu} = \alpha \eta_{\mu\nu} + m_{\mu\nu},$$

where $m_{\mu\nu} = -m_{\nu\mu}$. Whereas the pure trace results in an infinitesimal dilation, the antisymmetric part effects an infinitesimal rotation. The term in Eq. (3.5) that remains to be investigated is the quadratic term, and when we substitute it in Eq. (3.3), we get

$$c_{\mu\nu\rho} = \eta_{\mu\rho} b_\nu + \eta_{\mu\nu} b_\rho - \eta_{\nu\rho} b_\mu$$

with $b_\mu = (1/d)c^\sigma{}_{\sigma\mu}$, and the corresponding transformation is given by

$$x'^\mu = x^\mu + 2(b_\rho x^\rho)x^\mu - b^\mu x^2,$$

which is referred to as a *special conformal transformation*, or *SCT* for short. Finite transformations corresponding to the infinitesimal transformations just discussed are given in the following list:

$$\begin{aligned}
\text{Translation} &\rightarrow x'^\mu = x^\mu + a^\mu \\
\text{Dilation} &\rightarrow x'^\mu = \alpha x^\mu \\
\text{Rotation} &\rightarrow x'^\mu = M^\mu{}_\nu x^\nu \\
\text{SCT} &\rightarrow x'^\mu = \frac{x^\mu - b^\mu x^2}{1 - 2b_\rho x^\rho + b^2 x^2}
\end{aligned} \tag{3.6}$$

The first three exponentiations are familiar from the usual symmetry transformations in classical mechanics, but the last one is not. Its validity, however, could be seen from the fact that its infinitesimal version, which could be obtained by expanding the denominator for small b^μ, is precisely the one given above. The factor scaling the metric for an SCT is

$$\Lambda(x) = (1 - 2b_\rho x^\rho + b^2 x^2)^2,$$

and we also note that a finite SCT could be rewritten as

$$\frac{x'^\mu}{x'^2} = \frac{x^\mu}{x^2} - b^\mu.$$

What this set of relations says is that a finite SCT could be viewed as inversion of x^μ followed by the translation b^μ, which, in turn, is followed by yet another inversion (see Fig. 3.1).

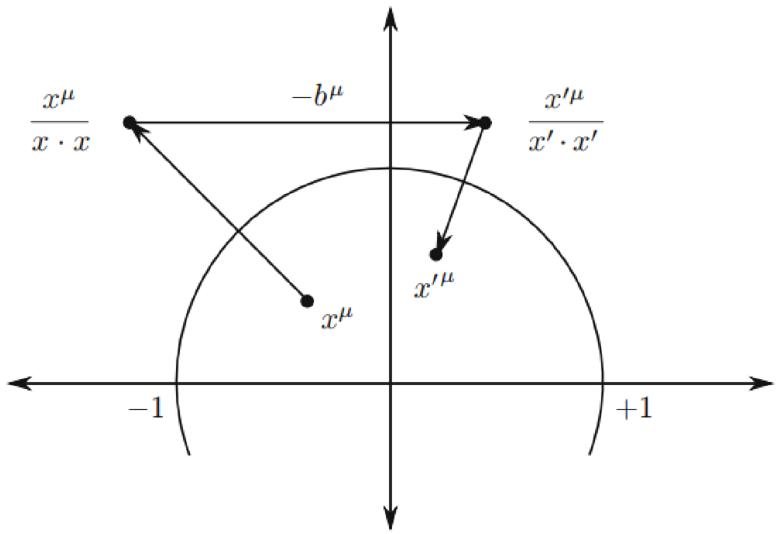


Figure 3.1: Special conformal transformation (SCT). An inversion is followed by the translation b^μ, which is followed by yet another inversion.

It is common to refer to a transformation via its *generator*, G_a, which is defined by the following infinitesimal transformation at the same point:

$$\varphi'(x) - \varphi(x) = -\imath\omega_a G_a \varphi(x). \tag{3.7}$$

The ω_a are the same parameters that were introduced in Eq. (1.5) of chapter 1. In fact, we could use Eq. (1.5) to do the following calculation:

$$\begin{aligned} \varphi'(x') &= \varphi(x) + \omega_a \frac{\partial \mathcal{F}}{\partial \omega_a}(x) \\ &= \varphi(x') - \omega_a \frac{\partial x^\mu}{\partial \omega_a} \partial_\mu \varphi(x') + \omega_a \frac{\partial \mathcal{F}}{\partial \omega_a}(x'). \end{aligned}$$

Comparing this result with Eq. (3.7) allows us to write

$$\imath G_a \varphi = \frac{\partial x^\mu}{\partial \omega_a} \partial_\mu \varphi - \frac{\partial \mathcal{F}}{\partial \omega_a}. \tag{3.8}$$

Now, if we assume for the moment that the fields themselves are not affected by any of the infinitesimal transformations (that is, $\mathcal{F}(\varphi) = \varphi$), then the generators for these transformations, found via Eq. (3.8), turn out to be the following:

$$\begin{aligned}
\text{Translation} &\rightarrow P_\mu = -\iota\partial_\mu \\
\text{Dilation} &\rightarrow D = -\iota x^\mu \partial_\mu \\
\text{Rotation} &\rightarrow L_{\mu\nu} = \iota(x_\mu\partial_\nu - x_\nu\partial_\mu) \\
\text{SCT} &\rightarrow K_\mu = -\iota(2x_\mu x^\nu \partial_\nu - x^2\partial_\mu)
\end{aligned} \tag{3.9}$$

Satisfying the following commutation relations, these generators define what is known as a *conformal algebra*.

$$\begin{aligned}
[D, P_\mu] &= \iota P_\mu; \\
[D, K_\mu] &= -\iota K_\mu; \\
[K_\mu, P_\nu] &= 2\iota(g_{\mu\nu}D - L_{\mu\nu}); \\
[K_\rho, L_{\mu\nu}] &= \iota(g_{\rho\mu}K_\nu - g_{\rho\nu}K_\mu); \\
[P_\rho, L_{\mu\nu}] &= \iota(g_{\rho\mu}P_\nu - g_{\rho\nu}P_\mu); \\
[L_{\mu\nu}, L_{\rho\sigma}] &= \iota(g_{\nu\rho}L_{\mu\sigma} + g_{\mu\sigma}L_{\nu\rho} - g_{\mu\rho}L_{\nu\sigma} - g_{\nu\sigma}L_{\mu\rho}).
\end{aligned} \tag{3.10}$$

It behooves us at this point to distinguish the *conformal algebra* formed by the generators above from the *conformal group* formed by the finite transformations in Eq. (3.6). Whereas the conformal group is the group comprised of globally defined, invertible, and finite conformal transformations, the conformal algebra is the Lie algebra corresponding to the conformal group.

To simplify the commutation relations above, we define the generators $J_{\mu\nu} = L_{\mu\nu}$, $J_{-1,\mu} = (1/2)(P_\mu - K_\mu)$, $J_{-1,0} = D$, and $J_{0,\mu} = (1/2)(P_\mu + K_\mu)$, where $J_{ab} = -J_{ba}$ and $a, b \in$

$\{-1, 0, 1, \ldots, d-1\}$. They satisfy the commutation relations of the Lie algebra $SO(d+1,1)$:

$$[J_{ab}, J_{cd}] = \iota(g_{ad}J_{bc} + g_{bc}J_{ad} - g_{ac}J_{bd} - g_{bd}J_{ac}).$$

Therefore, the conformal group in d spacetime dimensions is isomorphic to the group $SO(d+1,1)$, having d (from P_μ) $+ 1$ (from D) $+ d(d-1)/2$ (from $L_{\mu\nu}$) $+ d$ (from K_μ) $= (d+1)(d+2)/2$ generators.

3.2 Conformal symmetry in classical field theory

In the classical regime, a field theory is said to have *conformal symmetry* if its action is invariant under conformal transformations, and a discussion of conformal symmetry in this regime entails studying how conformal transformations go about affecting fields. Essentially, for an infinitesimal conformal transformation parameterized by ω_g, we seek a representation, T_g, such that a field, φ, transforms as

$$\varphi'(0) = (1 - \iota\omega_g T_g)\varphi(0).$$

The generator T_g must then be added to the spacetime part given in Eq. (3.9) to obtain the full generator of symmetry as given in Eq. (3.8). The trick we use is to consider a subgroup of the full conformal group that leaves the origin ($x = 0$) fixed, that is, the subgroup generated by rotations, dilations, and SCTs. Finding a representation of this subgroup at the origin, we use the commutation relations in Eq. (3.10) to translate the generators of the representation to other points in spacetime and get the full conformal group there.

For instance, using $\tilde{\Delta}$, $S_{\mu\nu}$, and κ_μ to denote the respective values of the generators D, $L_{\mu\nu}$, and K_μ at the origin, we know that they must form a representation of the reduced conformal algebra that satisfies the following:

$$
\begin{aligned}
[\tilde{\Delta}, S_{\mu\nu}] &= 0; \\
[\tilde{\Delta}, \kappa_\mu] &= -\imath\kappa_\mu; \\
[\kappa_\nu, \kappa_\mu] &= 0; \\
[\kappa_\rho, S_{\mu\nu}] &= \imath(\eta_{\rho\mu}\kappa_\nu - \eta_{\rho\nu}\kappa_\mu); \\
[S_{\mu\nu}, S_{\rho\sigma}] &= \imath(\eta_{\nu\rho}S_{\mu\sigma} + \eta_{\mu\sigma}S_{\nu\rho} - \eta_{\mu\rho}S_{\nu\sigma} - \eta_{\nu\sigma}S_{\mu\rho}).
\end{aligned}
$$

Now, we translate these generators using the translation operator, the commutation relations in Eq. (3.10), and the famous Baker-Campbell-Hausdorff formula:

$$e^{\imath x^\rho P_\rho} D e^{-\imath x^\rho P_\rho} = D + x^\nu P_\nu,$$

$$e^{\imath x^\rho P_\rho} L_{\mu\nu} e^{-\imath x^\rho P_\rho} = S_{\mu\nu} - x_\mu P_\nu + x_\nu P_\mu,$$

and

$$e^{\imath x^\rho P_\rho} K_\mu e^{-\imath x^\rho P_\rho} = K_\mu + 2x_\mu D - 2x^\nu L_{\mu\nu} + 2x_\mu(x^\nu P_\nu) - x^2 P_\mu.$$

These translated generators tell us how the field would be affected at points of spacetime different from the origin, for we get some new transformation rules, that is,

$$D\varphi(x) = (-\imath x^\nu \partial_\nu + \tilde{\Delta})\varphi(x),$$

$$L_{\mu\nu}\varphi(x) = \imath(x_\mu\partial_\nu - x_\nu\partial_\mu)\varphi(x) + S_{\mu\nu}\varphi(x),$$

and

$$K_\mu\varphi(x) = (\kappa_\mu + 2x_\mu\tilde{\Delta} - 2x^\nu S_{\mu\nu} - 2\imath x_\mu x^\nu\partial_\nu + \imath x^2\partial_\mu)\varphi(x).$$

Now, if we require that φ belong to an irreducible representation of the Lorentz group, then according to Schur's lemma, a generator commuting with all the $S_{\mu\nu}$ must be a multiple of the identity, and with the commutation relations for $\tilde{\Delta}$, $S_{\mu\nu}$, and κ_μ, we could conclude that it is $\tilde{\Delta}$ that should be proportional to the identity. The commutation relations also show that in this particular case, all the κ_μ vanish. Setting $\tilde{\Delta}$ equal to $-\iota\Delta$ then, where Δ is what is known as the *conformal dimension* of the field, we could show the result (a result which we leave the proof of) that a (spinless, or $S_{\mu\nu} = 0$) field, φ, transforms under a conformal tranformation, $x \to x'$, as

$$\varphi'(x') = \left|\frac{\partial x'}{\partial x}\right|^{-\Delta/d} \varphi(x), \tag{3.11}$$

where $|\partial x'/\partial x|$ is the Jacobian. A field transforming in this way, as we discuss in detail later too, is called a *quasi-primary field*. Having explicit forms of the generators and the way they act on the field φ in this case, we could go on to construct an action that is invariant under the transformations that the generators effect, namely the conformal transformations.

3.3 Conformal symmetry in quantum field theory

To discuss conformal symmetry in the quantum regime, we look at the consequences it has for 2- and 3-point correlators of quasi-primary fields. We start by considering the 2-point correlator

$$\langle \varphi_1(x_1)\varphi_2(x_2)\rangle = \lim_{\varepsilon\to 0}\frac{1}{Z}\int [d\varphi]\varphi_1(x_1)\varphi_2(x_2)\exp(\iota S_\varepsilon[\varphi]),$$

where φ_1 and φ_2 are quasi-primary fields, φ stands for the set of all the fields in the theory, Z represents the denominator of the expression in Eq. (2.2) but for the case of fields, and the action $S[\varphi]$ is assumed to be invariant under conformal transformations. We would also assume conformal invariance of the integration measure. Then, under a conformal transformation, the correlator must transform as

$$\left\langle \varphi_1(x_1)\varphi_2(x_2) \right\rangle = \left|\frac{\partial x'}{\partial x}\right|_{x=x_1}^{\Delta_1/d} \left|\frac{\partial x'}{\partial x}\right|_{x=x_2}^{\Delta_2/d} \left\langle \varphi_1(x_1')\varphi_2(x_2') \right\rangle,$$

and specializing to a dilation, $x \to x' = \lambda x$, we see that

$$\left\langle \varphi_1(x_1)\varphi_2(x_2) \right\rangle = \lambda^{\Delta_1}\lambda^{\Delta_2} \left\langle \varphi_1(\lambda x_1)\varphi_2(\lambda x_2) \right\rangle$$

since the Jacobian for the dilation in a d dimensional spacetime is λ^d. Rotational and translational symmetries require that

$$\left\langle \varphi_1(x_1)\varphi_2(x_2) \right\rangle = f(|x_1 - x_2|)$$

with $f(x) = \lambda^{\Delta_1+\Delta_2} f(\lambda x)$ to incorporate the symmetry under dilations. This thing prompts us to write

$$\left\langle \varphi_1(x_1)\varphi_2(x_2) \right\rangle = \frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}},$$

where C_{12} is a constant. Symmetry under SCTs still remains to be used. We note that for an SCT,

$$\left|\frac{\partial x'}{\partial x}\right| = \frac{1}{(1 - 2b_\rho x^\rho + b^2x^2)^d}$$

and the distance $\left|x_i - x_j\right|$ transforms as

$$\left|x_i' - x_j'\right| = \frac{\left|x_i - x_j\right|}{(1 - 2b_\rho x_i^\rho + b^2x_i^2)^{1/2}(1 - 2b_\rho x_j^\rho + b^2x_j^2)^{1/2}}.$$

Then, keeping the expression obtained for $\left\langle \varphi_1(x_1)\varphi_2(x_2) \right\rangle$ above in sight, we conclude that under an SCT,

$$\frac{C_{12}}{|x_1 - x_2|^{\Delta_1+\Delta_2}} = \frac{C_{12}}{\gamma_1^{\Delta_1}\gamma_2^{\Delta_2}} \frac{(\gamma_1\gamma_2)^{\Delta_1+\Delta_2}}{|x_1 - x_2|^{\Delta_1+\Delta_2}}$$

with

$$\gamma_i = 1 - 2b_\rho x_i^\rho + b^2 x_i^2,$$

a constraint which is identically satisfied only if $\Delta_1 = \Delta_2$. In other words, only if two quasi-primary have the same conformal dimension are they correlated, that is,

$$\langle \varphi_1(x_1)\varphi_2(x_2)\rangle = \begin{cases} C_{12}/|x_1 - x_2|^{2\Delta_1}, & \Delta_1 = \Delta_2, \\ 0, & \Delta_1 \neq \Delta_2. \end{cases} \tag{3.12}$$

We could perform a similar analysis for a 3-point correlator as well, but we quote the result we get without giving the details:

$$\left\langle \varphi_1(x_1)\varphi_2(x_2)\varphi_3(x_3)\right\rangle = \frac{C_{123}}{x_{12}^{\Delta_1+\Delta_2-\Delta_3} x_{23}^{\Delta_2+\Delta_3-\Delta_1} x_{13}^{\Delta_3+\Delta_1-\Delta_2}},$$

where C_{123} is a constant. The important thing to note here is that we can easily work out the structure of different correlators using arguments based on conformal symmetries solely, or without referring to any Lagrangians and actions, a testimony to the ease that CFTs could afford.

3.4 Conformal field theory in $d = 2$ dimensions

The formulation of CFTs in 2 dimensions has developed to a particularly mature state over the last 30 years, impacting both physics and mathematics along the way. Therefore, they could be considered prototypical examples of a valuable interplay between the two disciplines.

Even though 2-dimensional CFTs are examples of quantum field theories, they are different from the ordinary QFTs in 4 dimensions, for they could be defined and solved in an abstract way via operator algebras and their representation theory. What makes this thing possible is that the algebra of infinitesimal conformal transformations in 2 dimensions is infinite-dimensional and thus highly constraining. We begin our study of 2-dimensional CFTs by going back to Eq. (3.2) and seeing that when $d = 2$, it implies the following.

$$\begin{aligned}\partial_0 \varepsilon_0 &= \partial_1 \varepsilon_1; \\ \partial_0 \varepsilon_1 &= -\partial_1 \varepsilon_0.\end{aligned} \tag{3.13}$$

As might be evident, we are working with the Euclidean metric in flat space here, but we would soon address the case of the Minkowski metric as well. What is important to note though is the fact that Eq. (3.13) has the well-known Cauchy-Riemann equations, which prompt us to view ε_0 and ε_1 as the real and imaginary parts, respectively, of a holomorphic function defined on some open set and introduce complex variables via the following relations:

$$\begin{gathered} z = x^0 + \iota x^1;\ \bar{z} = x^0 - \iota x^1 \\ \varepsilon = \varepsilon^0 + \iota \varepsilon^1;\ \bar{\varepsilon} = \varepsilon^0 - \iota \varepsilon^1 \\ \partial_z = \frac{1}{2}(\partial_0 - \iota \partial_1);\ \partial_{\bar{z}} = \frac{1}{2}(\partial_0 + \iota \partial_1) \end{gathered} \tag{3.14}$$

An important question that pops up regards the status of z and $\bar{z}$: whether they should be treated as being independent. The proper approach to doing the coordinate transformation above is to first extend the range of the coordinates x^0 and x^1 to incorporate the whole complex plane. The coordinate transformation given then is simply a change of variables, and $\bar{z}$ is not the complex conjugate of z; instead, it is an independent coordinate. What must be remembered at the end of the day, however, is that physical space is the 2-dimensional subset of this extension, and it is defined by the identification of $\bar{z}$ with z^*.

Since $\varepsilon(z)$ is holomorphic, $f(z) = z + \varepsilon(z)$ too is holomorphic, and we could view $f(z)$ as effecting the infinitesimal conformal transformation $z \rightarrow f(z)$; similarly, since $\bar{\varepsilon}(\bar{z})$ is antiholomorphic,

$\bar{f}(\bar{z}) = \bar{z} + \bar{\varepsilon}(\bar{z})$ too is antiholomorphic, and we could view $\bar{f}(\bar{z})$ as effecting the infinitesimal conformal transformation $\bar{z} \to \bar{f}(\bar{z})$. As a result of these transformations, the metric transforms as

$$ds^2 = (dx^0)^2 + (dx^1)^2 = dzd\bar{z} \to \frac{\partial f}{\partial z}\frac{\partial \bar{f}}{\partial \bar{z}} dzd\bar{z},$$

which allows us to see $(\partial f/\partial z)(\partial \bar{f}/\partial \bar{z}) = |\partial f/\partial z|^2$ as the scaling factor in Eq. (3.1) and tells us that the transformation $z \to f(z)$, $\bar{z} \to \bar{f}(\bar{z})$ is indeed conformal. We have done our analysis for 2-dimensional Euclidean space only, but even if we had worked with 2-dimensional Minkowski space, our results would have been the same. In the latter case, the so-called light cone coordinates are used: $u = -t+x$ and $v = t+x$, where t denotes time and x denotes space. In these coordinates, the metric is ds^2 $(= -dt^2 + dx^2)$ $= dudv$, and it transforms as

$$ds^2 = dudv \to \frac{\partial f}{\partial u}\frac{\partial g}{\partial v} dudv$$

under the infinitesimal conformal transformation $u \to f(u)$, $v \to g(v)$; that is, it transforms in the same way as the metric expressed in terms of dz and $d\bar{z}$ transforms under an infinitesimal conformal transformation. The algebra of infinitesimal conformal transformations is thus again infinite-dimensional.

We now know that for an infinitesimal conformal transformation in two dimensions, $\varepsilon(z)$ has to be holomorphic on some open set. We could, however, assume that in general, it is a meromorphic function that has some isolated singularities outside this open set. Same could be expected of $\bar{\varepsilon}(\bar{z})$, and then, an infinitesimal conformal transformation in two dimensions could be written via

Laurent expansions around $z = 0$ and $\bar{z} = 0$ in the following manner.

$$z' = z + \varepsilon(z) = z + \sum_{n\in\mathbb{Z}} \varepsilon_n(-z^{n+1});$$
$$\bar{z}' = \bar{z} + \bar{\varepsilon}(\bar{z}) = \bar{z} + \sum_{n\in\mathbb{Z}} \bar{\varepsilon}_n(-\bar{z}^{n+1}).$$

The generators corresponding to the transformation for a particular n are

$$l_n = -z^{n+1}\partial_z$$

and

$$\bar{l}_n = -\bar{z}^{n+1}\partial_{\bar{z}},$$

and since $n \in \mathbb{Z}$, the total number of independent infinitesimal conformal transformations is infinite. We have thus verified a fact that we set forth right at the outset, namely that the algebra of infinitesimal conformal transformations in 2 dimensions is infinite-dimensional. This fact is specific to CFTs in 2 dimensions and, as we would see, has significant consequences.

To determine the algebra just mentioned, we would work out the commutation relations involving the generators l_n and $\bar{l}_n$.

$$\begin{aligned}
[l_m, l_n] &= z^{m+1}\partial_z(z^{n+1}\partial_z) - z^{n+1}\partial_z(z^{m+1}\partial_z) \\
&= (n+1)z^{m+n+1}\partial_z - (m+1)z^{m+n+1}\partial_z \\
&= -(m-n)z^{m+n+1}\partial_z \\
&= (m-n)l_{m+n}; \\
[\bar{l}_m, \bar{l}_n] &= (m-n)\bar{l}_{m+n}; \\
[l_m, \bar{l}_n] &= 0.
\end{aligned} \tag{3.15}$$

The respective sets of the generators l_n and $\bar{l}_n$ together define what is known as the *Witt algebra*, the conformal algebra of the generators that bring about infinitesimal conformal transformations in 2 dimensions. The most important observation here remains that this algebra is infinite-dimensional.

3.4.1 The conformal group—$d = 2$

Focusing on the part of the Witt algebra that is formed by the l_n, we see that these generators are not defined everywhere on $\mathbb{C}$; $z = 0$, for instance, is a point of ambiguity for l_{-2}. Also, to determine the conformal group, it turns out to be necessary to work not on $\mathbb{C}$ but on $\mathbb{C} \cup \{\infty\}$. $z = \infty$ then becomes another point of ambiguity.

For $z = 0$, we easily observe that the $l_n = -z^{n+1}\partial_z$ are nonsingular at $z = 0$ only if $n \geq -1$. In order to investigate $z = \infty$, however, we first need to do a change of variable via $z = -1/w$, which gives

$$l_n = -\left(-\frac{1}{w}\right)^{n-1}\partial_w,$$

and then let w approach 0. Upon doing so, we discover that the $l_n = -(-1/w)^{n-1}\partial_w$ are nonsingular at $w = 0$ (or $z = \infty$) only if $n \leq 1$. Putting these results together, we could conclude that globally defined conformal transformations, or those comprising the conformal group, are generated by l_{-1}, l_0, and l_1. This conclusion is what allows us to determine the conformal group when $d = 2$.

l_{-1}, right from the definition $l_{-1} = -\partial_z$, could be seen as being the generator of translations: $z \to z + b$. As to l_0, it equals $-z\partial_z$ and generates transformations of the kind $z \to az$, where $a \in \mathbb{C}$. These transformations could easily be identified as being rotations and dilations. Finally, l_1, which is $-z^2\partial_z$, generates SCTs, which could be, as explained before (see Fig. 3.1), viewed as translations of $w = -1/z$. In fact, $cl_1 z = -cz^2$ could be obtained from $z \to z/(cz + 1)$ (which is equivalent to $w \to w - c$) by expansion for small c. Collecting all the observations, we could say that l_{-1}, l_0, and l_1 generate transformations of the form

$$z \to \frac{az+b}{cz+d}, \tag{3.16}$$

where $a, b, c, d \in \mathbb{C}$. This form represents the famous Mobius transformations, and for them to be invertible, $ad - bc \neq 0$. With this final constraint imposed, we may conclude that the conformal group for $\mathbb{C} \cup \{\infty\}$ is isomorphic to the Mobius group.

Before ending this section, however, we present a little appendage, which is related to the Witt algebra and would prove useful later. The Witt algebra admits something known as a *central extension*, which could be described as being an extension of its commutation relations. In general, the central extension $\tilde{\eth} = \eth \oplus \mathbb{C}$ of the Lie algebra $\eth$ by $\mathbb{C}$ is characterized by the following commutation relations:

$$[\tilde{x}, \tilde{y}]_{\tilde{\eth}} = [x, y]_{\eth} + cp(x, y),$$

$$[\tilde{x}, c]_{\tilde{\eth}} = 0,$$

and

$$\left[c, c'\right]_{\tilde{\eth}} = 0,$$

where $\tilde{x}, \tilde{y} \in \tilde{\eth}$, $x, y \in \eth$, and $c, c' \in \mathbb{C}$. Also, $p : \eth \times \eth \to \mathbb{C}$ here is bilinear. In our applying this extension to the Witt algebra, we would denote the l_n in the extended algebra by the L_n and write their commutation relations as

$$[L_m, L_n] = (m-n)L_{m+n} + cp(m, n). \tag{3.17}$$

It goes without saying that the same could be done for the $\bar{l}_n$ too. In order to determine the precise form of $p(m, n)$, we begin by noticing that $p(m, n) = -p(n, m)$ must be true to respect the antisymmetry of the Lie bracket. Also, it is always possible to make $p(1,-1)$ and $p(n,0)$ vanish by the redefinitions

$$L_0' = L_0 + \frac{cp(1,-1)}{2}$$

and

$$L_n' = L_n + \frac{cp(n,0)}{n},$$

where the second one holds for $n \neq 0$. In fact, working some commutators out using the new operators shows that $p(1,-1)$ and $p(n,0)$ indeed vanish:

$$\left[L_1', L_{-1}'\right] = 2L_0 + cp(1,-1) = 2L_0',$$

and

$$\left[L_n', L_0'\right] = nL_n + cp(n,0) = nL_n'.$$

Then, we simplify a particular Jacobi identity:

$$\begin{aligned}
0 &= \left[\left[L_m, L_n\right], L_0\right] + \left[\left[L_n, L_0\right], L_m\right] + \left[\left[L_0, L_m\right], L_n\right] \\
&= (m-n)cp(m+n,0) + ncp(n,m) - mcp(m,n), \\
&= (m+n)p(m,n),
\end{aligned}$$

which tells us that $p(m,n) = 0$ if $n \neq -m$. Hence, the only nonvanishing central extensions are given by $p(n,-n)$ for $|n| \geq 2$. Finally, we evaluate another Jacobi identity, which leads us to a recursion relation for $p(n,-n)$:

$$
\begin{aligned}
0 &= \left[\left[L_{-n+1}, L_n\right], L_{-1}\right] + \left[\left[L_n, L_{-1}\right], L_{-n+1}\right] + \left[\left[L_{-1}, L_{-n+1}\right], L_n\right] \\
&= \frac{n+1}{n-2} p(n-1, -n+1) \\
&= \frac{1}{2}\begin{pmatrix} n+1 \\ 3 \end{pmatrix} \\
&= \frac{1}{12}(n+1)n(n-1),
\end{aligned}
$$

where we introduced the normalization $p(2,-2) = 1/2$ in the penultimate step. Having found the explicit form for $p(m,n)$, we could summarize whatever we have said about the central extension by saying that the central extension of the Witt algebra is called the *Virasoro algebra* and could be written as

$$[L_m, L_n] = (m-n)L_{m+n} + \frac{c}{12}(m^3 - m)\delta_{m+n,0}. \tag{3.18}$$

c here is what is known as the *central charge*.

3.4.2 Primary fields

This section is meant to introduce some important concepts as regards 2-dimensional CFTs. First, we recall that in our scheme of things, z and $\bar{z}$ are independent variables, something which our having two commuting sets of operators in the Witt algebra (namely, the l_n and the $\bar{l}_n$) also attests to. What this thing implies for our fields is that

$$\varphi(x^0, x^1) \;\rightarrow\; \varphi(z, \bar{z}).$$

However, as we said earlier, for physical space, we would have to identify $\bar{z}$ with z^*. Fields depending only on z are called *chiral*, and those depending only on $\bar{z}$ are called *antichiral*. Terms holomorphic and antiholomorphic are also respectively used to differentiate them from each other.

Extremely important here is to introduce the notions of conformal dimensions and primary fields too. A field transforming as

$$\varphi(z,\bar{z}) \;\rightarrow\; \varphi'(z',\bar{z}') = \left(\frac{\partial f}{\partial z}\right)^{h} \left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} \varphi\left(f(z),\bar{f}(\bar{z})\right) \tag{3.19}$$

under the conformal transformation $z \rightarrow f(z), \bar{z} \rightarrow \bar{f}(\bar{z})$ is referred to as a *primary field* with *conformal dimensions* $(h,\bar{h})$. If Eq. (3.19) holds for the Mobius transformations only, then φ is referred to as a *quasi-primary field*. It is also important to note that not all fields in 2-dimensional CFTs are primary or quasi-primary; those that are not are known as *secondary fields*. We do an important calculation for primary fields here, for it would help us establish some important ideas in the next section as well.

If we perform the conformal transformation $z \rightarrow f(z) = z + \varepsilon(z), \bar{z} \rightarrow \bar{f}(\bar{z}) = \bar{z} + \bar{\varepsilon}(\bar{z})$ (with $|\varepsilon(z)|, |\bar{\varepsilon}(\bar{z})| \ll 1$), then up to first order in $\varepsilon(z)$ and $\bar{\varepsilon}(\bar{z})$, we have

$$\left(\frac{\partial f}{\partial z}\right)^{h} = 1 + h\partial_z\varepsilon(z) + \mathcal{O}(\varepsilon^2),$$

$$\left(\frac{\partial \bar{f}}{\partial \bar{z}}\right)^{\bar{h}} = 1 + \bar{h}\partial_{\bar{z}}\bar{\varepsilon}(\bar{z}) + \mathcal{O}(\bar{\varepsilon}^2),$$

and

$$\varphi(z+\varepsilon(z),\bar{z}+\bar{\varepsilon}(\bar{z})) = \varphi(z,\bar{z}) + \varepsilon(z)\partial_z\varphi(z,\bar{z}) + \bar{\varepsilon}(\bar{z})\partial_{\bar{z}}\varphi(z,\bar{z}) + \mathcal{O}(\varepsilon^2,\bar{\varepsilon}^2).$$

Using these results in Eq. (3.19) and sticking to first order, we find that

$$\varphi(z,\bar{z}) \;\rightarrow\; \varphi(z,\bar{z}) + (h\partial_z\varepsilon + \varepsilon\partial_z + \bar{h}\partial_{\bar{z}}\bar{\varepsilon} + \bar{\varepsilon}\partial_{\bar{z}})\varphi(z,\bar{z}). \tag{3.20}$$

This result would later be used to bring up the important notion of operator product expansion. For now, it is instructive to investigate another result—which we state without proof—and its important consequences: the energy-momentum tensor $T_{\mu\nu}$ of a 2-dimensional CFT is traceless, that is,

$$T^{\mu}_{\mu} = 0 \tag{3.21}$$

for 2-dimensional CFTs. We begin exploring the implications of this thing by using Eq. (3.14) to go from the real to the complex coordinates, thereby obtaining

$$T_{zz} = \frac{1}{4}(T_{00} - 2\iota T_{10} - T_{11}),$$

$$T_{\bar{z}\bar{z}} = \frac{1}{4}(T_{00} + 2\iota T_{10} - T_{11}),$$

and

$$T_{z\bar{z}} = T_{\bar{z}z} = \frac{1}{4}(T_{00} + T_{11}) = \frac{1}{4}T^{\mu}_{\mu} = 0.$$

Using the tracelessness condition in the first two relations as well, we find that $T_{zz} = (1/2)(T_{00} - \iota T_{10})$ and $T_{\bar{z}\bar{z}} = (1/2)(T_{00} + \iota T_{10})$. Using these relations with the fact that $\partial^{\mu} T_{\mu\nu} = 0$ then gives

$$\begin{aligned}
\partial_{\bar{z}} T_{zz} &= \frac{1}{4}(\partial_0 + \iota\partial_1)(T_{00} - \iota T_{10}) \\
&= \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + \iota\partial_1 T_{00} - \iota\partial_0 T_{10}) \\
&= \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} - \iota\partial_1 T_{11} - \iota\partial_0 T_{01}) \\
&= 0
\end{aligned}$$

and

$$
\begin{aligned}
\partial_z T_{\bar{z}\bar{z}} &= \frac{1}{4}(\partial_0 - \iota\partial_1)(T_{00} + \iota T_{10}) \\
&= \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} - \iota\partial_1 T_{00} + \iota\partial_0 T_{10}) \\
&= \frac{1}{4}(\partial_0 T_{00} + \partial_1 T_{10} + \iota\partial_1 T_{11} + \iota\partial_0 T_{01}) \\
&= 0.
\end{aligned}
$$

Clearly, we have established that $\partial_{\bar{z}} T_{zz} = 0$ and $\partial_z T_{\bar{z}\bar{z}} = 0$, meaning that the only nonzero components of the energy-momentum tensor are a chiral field and an antichiral field: $T_{zz}(z, \bar{z}) = T(z)$ and $T_{\bar{z}\bar{z}} = \bar{T}(\bar{z})$, respectively.

3.4.3 Radial quantization and OPE

We have been doing 2-dimensional CFTs with the Euclidean metric, using x^0 to denote time and x^1 to denote space. What is often done in such theories is that space is compactified on a circle of radius R, which is mostly set equal to 1. In other words, x^1 is identified with $x^1 + 2\pi R = x^1 + 2\pi$. The CFT obtained as a result then is defined on a cylinder of infinite length, and we could define a single coordinate, w, to describe points on it.

$$w = x^0 + \iota x^1;\ w \sim w + 2\pi\iota.$$

The identification $w \sim w + 2\pi\iota$ follows from the compactification done above. Having defined the theory in this manner, we move from our cylinder to the complex plane through the mapping

$$z = e^w = e^{x^0 + \iota x^1}, \tag{3.22}$$

which allows us to use the complex coordinate z in lieu of w (see Fig. 3.2). A time translation on the cylinder, $x^0 \to x^0 + a$, gets mapped to a dilation of the kind $z \to \exp(a)z$, and a space translation on the cylinder, $x^1 \to x^1 + b$, gets mapped to a rotation of the kind $z \to \exp(\imath b)z$.

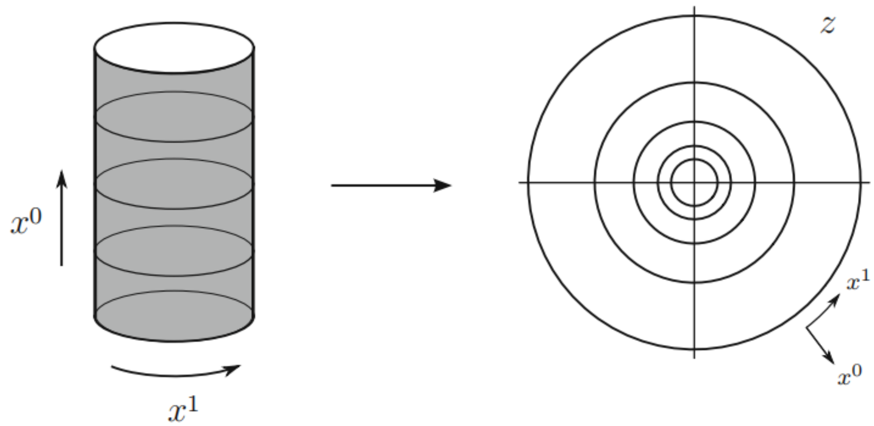


Figure 3.2: Mapping the infinite cylinder to the complex plane. x^0 becomes yoked to dilations, and x^1 gets tethered to rotations.

If we now consider a primary field $\varphi(z, \bar{z})$ with conformal dimensions $(h, \bar{h})$ that we can perform a Laurent expansion for about $z_0 = \bar{z}_0 = 0$, then the expansion turns out to be

$$\varphi(z, \bar{z}) = \sum_{n, \bar{m} \in \mathbb{Z}} z^{-n-h} \bar{z}^{-\bar{m}-\bar{h}} \varphi_{n, \bar{m}}, \tag{3.23}$$

where the factors of h and $\bar{h}$ could be explained by Eq. (3.19) and the mapping given in Eq. (3.22). The so-called *radial quantization* of the field is achieved by promoting the Laurent modes $\varphi_{n,\bar{m}}$ to operators. One could then notice that the mapping in Eq. (3.22) maps the infinite past on the cylinder, $x^0 = -\infty$, to $z = \bar{z} = 0$, a fact which allows us to define an *asymptotic in-state* as

$$|\varphi\rangle = \lim_{z, \bar{z} \to 0} \hat{\varphi}(z, \bar{z}) \, |0\rangle \,,$$

but in order for this state to be a "well-behaved" one, all singularities at $z = \bar{z} = 0$ should be avoided. Therefore, we require that $\hat{\varphi}_{n,\bar{m}} \left|0\right\rangle = 0$ for $n > -h$ or $\bar{m} > -\bar{h}$. This requirement, with the mode expansion in Eq. (3.23), gives

$$\left|\varphi\right\rangle = \hat{\varphi}_{-h,-\bar{h}} \left|0\right\rangle . \tag{3.24}$$

Having defined this in-state, we could define an *asymptotic out-state* as well. To do so, however, we need to establish for $\hat{\varphi}(z,\bar{z})$ the notion of Hermitian conjugation first. It must be remembered that the Euclidean coordinates are obtained from the Minkowski ones through the introduction of *complex time*, $t \to \imath t$, so the action of Hermitian conjugation on the Euclidean time x^0 is nontrivial, for $x^0 \to -x^0$. For $z = \exp\left(x^0 + \imath x^1\right)$ then, Hermitian conjugation amounts to $z \to 1/\bar{z}$. We thus define the Hermitian conjugate of $\hat{\varphi}(z,\bar{z})$ as

$$\hat{\varphi}^\dagger(z,\bar{z}) = \bar{z}^{-2h} z^{-2\bar{h}} \hat{\varphi}\left(\frac{1}{\bar{z}}, \frac{1}{z}\right),$$

and a Laurent expansion for it gives

$$\sum_{n,\bar{m}\in\mathbb{Z}} \bar{z}^{n-h} z^{\bar{m}-\bar{h}} \hat{\varphi}_{n,\bar{m}}.$$

Comparing this expansion with the Hermitian conjugate of the one in Eq. (3.23), we obtain

$$\left(\hat{\varphi}_{n,\bar{m}}\right)^\dagger = \hat{\varphi}_{-n,-\bar{m}}. \tag{3.25}$$

Finally, we could define an asymptotic out-state as

$$\left\langle\varphi\right| = \lim_{z,\bar{z}\to 0} \left\langle 0\right| \hat{\varphi}^\dagger(z,\bar{z}),$$

but if want to avoid all singularities at $z = \bar{z} = 0$, we need to require that $\langle 0| \hat{\varphi}_{n,\bar{m}} = 0$ for $n < h$ or $\bar{m} < \bar{h}$. This requirement then, with the mode expansion for the Hermitian conjugate of $\hat{\varphi}(z, \bar{z})$, gives

$$\langle \varphi| = \langle 0| \hat{\varphi}_{h,\bar{h}}. \tag{3.26}$$

Now, as we discussed in the context of Noether's theorem in section 1.1, field theories with continuous symmetries happen to have conserved charges. For a field theory in Euclidean coordinates, x^0 and x^1, the conserved charge may be written as

$$Q = \int dx^1 j_0,$$

where j_0 is what we earlier referred to as a current. It is well-known that a conserved charge serves as the generator of transformations for various quantities, and for some random quantity A, this fact may roughly be written as

$$\delta\hat{A} = \left[\hat{Q}, \hat{A}\right]$$

with the commutation relation evaluated at equal times. We must also note here that the integral for conserved charge above too was evaluated at some particular time, something implying that x^0 was a constant for that integral. Seen in the light of the coordinate change given in Eq. (3.22), this thing translates to the facts that $|z|$ is a constant for that integral and that it should become a contour integral after the coordinate change. Following the convention that contour integrals are always worked out in a counterclockwise manner, we could say that the most natural generalization of the conserved charge integral for complex coordinates is

$$Q = \frac{1}{2\pi\imath} \oint_C \left(dz T(z)\varepsilon(z) + d\bar{z}\bar{T}(\bar{z})\bar{\varepsilon}(\bar{z}) \right), \tag{3.27}$$

where we have used $T_{\mu\nu}\varepsilon^{\nu}$ in place of the current that one gets under conformal symmetries. As might easily be noted, we have started considering $\bar{z}$ and $\bar{\varepsilon}(\bar{z})$ as well, but it must be remembered that doing so is necessary for full generality, something requiring us to assume that x^0 and x^1 are not always real; for physical space, however, as said before, we would have to restrict x^0 and x^1 to real values only.

In view of the commutation relation we wrote, we could now use Eq. (3.27) to write the infinitesimal change generated in the field φ by the conserved charge Q:

$$\delta_{\varepsilon,\bar{\varepsilon}}\hat{\varphi}(w,\bar{w}) = \frac{1}{2\pi\imath}\oint_C dz\left[\hat{T}(z)\varepsilon(z),\hat{\varphi}(w,\bar{w})\right] + \frac{1}{2\pi\imath}\oint_C d\bar{z}\left[\hat{\bar{T}}(\bar{z})\bar{\varepsilon}(\bar{z}),\hat{\varphi}(w,\bar{w})\right]. \tag{3.28}$$

This very relation is what brings us to the notion of *radial ordering*, for we do encounter an ordering ambiguity in the equation above: we have to decide whether w and $\bar{w}$ are inside the contour C since $|w|$ and $|\bar{w}|$ give a measure of time and $\delta\hat{A} = \left[\hat{Q},\hat{A}\right]$ is evaluated at equal times. As might be remembered from chapter 2, the ordering of operators in correlators is set by the time ordering operator. The order of operators in Eq. (3.28) would be set in precisely the same way, but since the change to complex coordinates linked time translations to dilations, time ordering would be implemented through what was just called radial ordering. We define the radial ordering of two operators, $\hat{A}(z)$ and $\hat{B}(w)$, in the following way:

$$R\left(\hat{A}(z)\hat{B}(w)\right) = \begin{cases}\hat{A}(z)\hat{B}(w), & |z| > |w|,\\ \hat{B}(w)\hat{A}(z), & |w| > |z|.\end{cases} \tag{3.29}$$

With this definition, we know that the correct way to interpret the commutators in Eq. (3.28) is given by

$$\oint dz \left[\hat{A}(z), \hat{B}(w)\right] = \oint_{|z|>|w|} dz\hat{A}(z)\hat{B}(w) - \oint_{|z|<|w|} dz\hat{B}(w)\hat{A}(z)$$
$$= \oint_{C(w)} dzR\left(\hat{A}(z)\hat{B}(w)\right),$$

which allows us to rewrite Eq. (3.28) as

$$\delta_{\varepsilon,\bar{\varepsilon}}\hat{\varphi}(w,\bar{w}) = \frac{1}{2\pi\iota}\oint_{C(w)} dz\varepsilon(z)R\left(\hat{T}(z)\hat{\varphi}(w,\bar{w})\right) + \text{Antichiral.}$$

However, we did calculate the infinitesimal change in the field φ in Eq. (3.20), and to get that very expression from the contour integrals we have now, we realize that we must have

$$R\left(\hat{T}(z)\hat{\varphi}(w,\bar{w})\right) = \frac{h}{(z-w)^2}\hat{\varphi}(w,\bar{w}) + \frac{1}{z-w}\partial_w\hat{\varphi}(w,\bar{w}) + \ldots$$

and

$$R\left(\hat{\bar{T}}(\bar{z})\hat{\varphi}(w,\bar{w})\right) = \frac{\bar{h}}{(\bar{z}-\bar{w})^2}\hat{\varphi}(w,\bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\hat{\varphi}(w,\bar{w}) + \ldots,$$

where ... represent nonsingular terms. Just eyeballing the insertion of these "expansions" into the expression we found for the infinitesimal change shows that they are correct, and we go on to call them *operator product expansions*, or *OPEs* for short, which define an algebraic product structure on quantum fields. Since the OPEs above specifically correspond to the infinitesimal change in a primary field with conformal dimensions $(h, \bar{h})$, we could use them to actually redefine such a field, the new definition being the following: a primary field with conformal dimensions $(h, \bar{h})$ is one that has the OPEs given above with the energy-momentum tensor.

We could also go about calculating the OPE of the chiral part of the energy-momentum tensor with itself:

$$R\left(\hat{T}(z)\hat{T}(w)\right) = \frac{c/2}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w\hat{T}(w)}{z-w} + \ldots,$$

where c is what we referred to as the central charge while discussing the Virasoro algebra in section 3.4.1. This OPE could be shown to be the correct one via some calculations, which begin by Laurent expanding the chiral part of the energy-momentum tensor in the following way.

$$\hat{T}(z) = \sum_{n\in\mathbb{Z}} z^{-n-2}\hat{L}_n;\ \hat{L}_n = \frac{1}{2\pi\imath}\oint dz z^{n+1}\hat{T}(z).$$

If we use this expansion in Eq. (3.27) with the particular conformal transformation $\varepsilon(z) = -\varepsilon_n z^{n+1}$, we get

$$Q_n = \oint \frac{dz}{2\pi\imath}\hat{T}(z)(-\varepsilon_n z^{n+1}) = -\varepsilon_n \sum_{m\in\mathbb{Z}} \oint \frac{dz}{2\pi\imath}\hat{L}_m z^{n-m-1} = -\varepsilon_n \hat{L}_n,$$

which clearly shows that the Laurent modes $\hat{L}_n$ of $\hat{T}(z)$ could be identified with the generators of infinitesimal conformal transformations. They should, as a result, satisfy the Virasoro algebra we introduced in section 3.4.1. Using the expression for them in terms of the chiral part of the energy-momentum tensor (see above), we could go about evaluating $\left[\hat{L}_m, \hat{L}_n\right]$:

$$\begin{aligned}
\left[\hat{L}_m, \hat{L}_n\right] &= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+1} \oint_{C(w)} \frac{dz}{2\pi\imath} z^{m+1} R\left(\hat{T}(z)\hat{T}(w)\right) \\
&= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+1} \oint_{C(w)} \frac{dz}{2\pi\imath} z^{m+1} \left(\frac{c/2}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w \hat{T}(w)}{z-w}\right) \\
&= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+1} \left[\frac{c(m+1)m(m-1)w^{m-2}}{2(3!)} + 2(m+1)w^m \hat{T}(w) + w^{m+1}\partial_w \hat{T}(w)\right] \\
&= (m-n)\hat{L}_{m+n} + \frac{c}{12}\left(m^3 - m\right)\delta_{m,-n},
\end{aligned}$$

where we employed integration by parts to deal with the last term in the penultimate line. Since the L_n clearly satisfy the Virasoro algebra, the OPE of $T(z)$ with itself that we gave above is right. The calculation essentially shows that the singular part of this OPE is equivalent to the Virasoro algebra formed by the L_n.

Before closing this section off, we offer one more particularly important result, namely that $\left[\hat{L}_m, \hat{\varphi}_n\right] = [(h-1)m - n]\hat{\varphi}_{m+n}$. This result could easily be worked out by going along the lines of the calculation above:

$$\begin{aligned}
\left[\hat{L}_m, \hat{\varphi}_n\right] &= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+h-1} \oint_{C(w)} \frac{dz}{2\pi\imath} z^{m+1} R\left(\hat{T}(z)\hat{\varphi}(w)\right) \\
&= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+h-1} \oint_{C(w)} \frac{dz}{2\pi\imath} z^{m+1} \left(\frac{h}{(z-w)^2}\hat{\varphi}(w) + \frac{1}{z-w}\partial_w \hat{\varphi}(w)\right) \\
&= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+h-1} \left[h(m+1)w^m \hat{\varphi}(w) + w^{m+1}\partial_w \hat{\varphi}(w)\right] \\
&= [(h-1)m - n]\hat{\varphi}_{m+n}.
\end{aligned}$$

3.4.4 Normal ordering

The idea of normal ordering was first introduced in the context of Wick's theorem in section 2.4 and was described as an operation that brings all the creation operators to the left of all the annihilation operators. In this section, we revisit this operation but discuss it in the context of 2-dimensional CFTs.

Before we bring up the operation itself, however, we need to establish what the creation and the annihilation operators are for the case at hand. Recalling the discussion of the asymptotic in- and out-states in section 3.4.3, we note that

$$\hat{\varphi}_{n,\bar{m}} \left|0\right\rangle = 0$$

for $n > -h$ and $\bar{m} > -\bar{h}$. This thing clearly tells us that all $\hat{\varphi}_{n,\bar{m}}$ with $n > -h$ or $\bar{m} > -\bar{h}$ could be interpreted as annihilation operators. The rest would be the creation ones, and they could be required to create states with positive energy only. In summary, we could say that for a chiral primary field, φ, with the conformal dimension h, $\hat{\varphi}_n$ with $n > -h$ are the annihilation operators, and $\hat{\varphi}_n$ with $n \leq -h$ are the creation operators.

To proceed to the discussion of normal ordering now, we state a result without proof that we go on to simply verify: the nonsingular part of an OPE naturally gives rise to normal ordered products, a fact which could be expressed in the following manner:

$$R(\hat{\varphi}(z)\hat{\chi}(w)) = \text{Singular} + \sum_{n=0}^{\infty} \frac{(z-w)^n}{n!} \mathcal{N}(\hat{\chi}\partial^n \hat{\varphi})(w). \tag{3.30}$$

We verify this result in what follows for the case $n = 0$. In this case, we could easily use Eq. (3.30) and properties of contour integration to obtain the normal ordered product of two operators:

$$\mathcal{N}(\hat{\chi}\hat{\varphi})(w) = \oint_{C(w)} \frac{dz}{2\pi\imath} \frac{R(\hat{\varphi}(z)\hat{\chi}(w))}{z-w}.$$

However, we could also have simply Laurent expanded $\mathcal{N}(\hat{\chi}\hat{\varphi})$ as usual, that is,

$$\mathcal{N}(\hat{\chi}\hat{\varphi})_n = \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+h^{\varphi}+h^{\chi}-1} \mathcal{N}(\hat{\chi}\hat{\varphi})(w),$$

where h^{φ} and h^{χ} are the respective conformal dimensions of $\hat{\varphi}$ and $\hat{\chi}$. Inserting in this result the expression we found for $\mathcal{N}(\hat{\chi}\hat{\varphi})(w)$ gives

$$\begin{aligned}
\mathcal{N}(\hat{\chi}\hat{\varphi})_n &= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+h^{\varphi}+h^{\chi}-1} \oint_{C(w)} \frac{dz}{2\pi\imath} \frac{R(\hat{\varphi}(z)\hat{\chi}(w))}{z-w} \\
&= \oint_{C(0)} \frac{dw}{2\pi\imath} w^{n+h^{\varphi}+h^{\chi}-1} \left(\oint_{|z|>|w|} \frac{\hat{\varphi}(z)\hat{\chi}(w)}{z-w} - \oint_{|z|<|w|} \frac{\hat{\chi}(w)\hat{\varphi}(z)}{z-w} \right).
\end{aligned}$$

Now, we separately evaluate the first integral in the parentheses, using in place of $\hat{\varphi}(z)$ and $\hat{\chi}(w)$

their respective Laurent expansions.

$$
\begin{aligned}
&\oint_{|z|>|w|} \frac{dz}{2\pi\imath} \frac{1}{z-w} \sum_{r,s} z^{-r-h^{\varphi}} w^{-s-h^{\chi}} \hat{\varphi}_r \hat{\chi}_s \\
&= \oint_{|z|>|w|} \frac{dz}{2\pi\imath} \frac{1}{z} \sum_{p\geq 0} \left(\frac{w}{z}\right)^p \sum_{r,s} z^{-r-h^{\varphi}} w^{-s-h^{\chi}} \hat{\varphi}_r \hat{\chi}_s \\
&= \oint_{|z|>|w|} \frac{dz}{2\pi\imath} \sum_{p\geq 0} \sum_{r,s} z^{-r-h^{\varphi}-p-1} w^{-s-h^{\chi}+p} \hat{\varphi}_r \hat{\chi}_s,
\end{aligned}
$$

where we used $1/(z-w) = (1/z)(1/(1-w/z))$ and the geometric series in the second line. Then, knowing that only the z^{-1} term contributes in the contour integration, we see that the integral over z sets r equal to $-h^{\varphi} - p$. When combined with the contour integral over w, this result produces the expression

$$
\begin{aligned}
&\oint \frac{dw}{2\pi\imath} \sum_{p\geq 0} \sum_{s} w^{-s-h^{\chi}+p+n+h^{\varphi}+h^{\chi}-1} \hat{\varphi}_{-h^{\varphi}-p} \hat{\chi}_s \\
&= \sum_{p\geq 0} \hat{\varphi}_{-h^{\varphi}-p} \hat{\chi}_{h^{\varphi}+n+p} \\
&= \sum_{k\leq -h^{\varphi}} \hat{\varphi}_k \hat{\chi}_{n-k}.
\end{aligned}
$$

As to the integral over w, we again know that only the w^{-1} term contributes. Therefore, to get rid of this integral, we set s equal to $p + n + h^{\varphi}$ in the second line. Introduction of the variable k then produces the third line. The second term in parentheses in the final expression on the previous page could be subjected to a similar analysis and turned into $\sum_{k>-h^{\varphi}} \hat{\chi}_{n-k} \hat{\varphi}_k$. A complete, simplified expression for $\mathcal{N}(\hat{\chi}\hat{\varphi})_n$ could thus be written as

$$
\mathcal{N}(\hat{\chi}\hat{\varphi})_n = \sum_{k>-h^{\varphi}} \hat{\chi}_{n-k} \hat{\varphi}_k + \sum_{k\leq -h^{\varphi}} \hat{\varphi}_k \hat{\chi}_{n-k}. \tag{3.31}
$$

As might be evident, the $\hat{\varphi}_k$ in the first sum are annihilation operators and are on the right whereas the $\hat{\varphi}_k$ in the second sum are creation operators and are on the left. Clearly then, the nonsingular part of an OPE contains normal ordered products.

3.4.5 Free bosonic field

Up till now, we have studied the generic structure of 2-dimensional CFTs and discussed some important aspects of them (algebras, primary fields, OPEs, and so on) without bringing in any Lagrangians and actions. The tools we have developed so far are particularly powerful for studying CFTs, and as we have mentioned every now and again, they are in some cases sufficient for a complete description of the dynamics. However, to make some contact with the usual approach to doing quantum field theories, we would, in this section and the next, consider examples of CFTs given in terms of Lagrangians and actions and discuss some of their properties and aspects in terms of the CFT jargon introduced in this chapter.

The example to be discussed in this section is that of a free massless scalar field, which happens to be a free bosonic field, defined on the infinite cylinder of section 3.4.4. The action for this field could be written in the following manner:

$$\begin{aligned} S &= \frac{1}{4\pi\kappa} \int dx^0 dx^1 \sqrt{|h|} h^{\mu\nu} \partial_\mu X \partial_\nu X \\ &= \frac{1}{4\pi\kappa} \int dx^0 dx^1 \left[(\partial_{x^0} X)^2 + (\partial_{x^1} X)^2 \right], \end{aligned} \tag{3.32}$$

where $h = \det h_{\mu\nu}$ with

$$h_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

and κ is some normalization constant. The similarity between this action and the Klein-Gordon field's (Eq. (1.10)) is quite striking; the only differences are that the scalar field here is defined on an infinite cylinder and that there is no mass term. The absence of a mass term indicates that nothing in the theory sets a scale, and hence, the theory is conformally invariant.

Before beginning our detailed study of this action, we follow the procedure of the previous section and map our cylinder to the complex plane via the mapping given in Eq. (3.22). The change of coordinates that doing so effects allows us to rewrite the action as

$$\begin{aligned} S &= \frac{1}{4\pi\kappa}\int dzd\bar{z}\sqrt{|g|}g^{\alpha\beta}\partial_\alpha X\partial_\beta X \\ &= \frac{1}{2\pi\kappa}\int dzd\bar{z}\partial_z X\partial_{\bar{z}} X, \end{aligned} \tag{3.33}$$

where $g = \det g_{\mu\nu}$ with

$$g_{\mu\nu} = \begin{bmatrix} 0 & 1/2z\bar{z} \\ 1/2z\bar{z} & 0 \end{bmatrix}.$$

Then, varying this action with respect to the field X yields

$$\begin{aligned} 0 &= \delta S \\ &= \frac{1}{2\pi\kappa}\int dzd\bar{z}(\delta(\partial_z X)\partial_{\bar{z}}X + \partial_z X\delta(\partial_{\bar{z}}Z)) \\ &= \frac{1}{2\pi\kappa}\int dzd\bar{z}(\partial_z(\delta X\partial_{\bar{z}}X) - \delta X\partial_z\partial_{\bar{z}}X + \partial_{\bar{z}}(\partial_z X\delta X) - \partial_{\bar{z}}\partial_z X\delta X) \\ &= -\frac{1}{2\pi\kappa}\int dzd\bar{z}\delta X(\partial_z\partial_{\bar{z}}X), \end{aligned}$$

which, required to hold for all δX, gives the following equation of motion:

$$\partial_z\partial_{\bar{z}}X = 0. \tag{3.34}$$

From this equation of motion, we can infer that

$$J(z) = \imath\partial_z X$$

is a chiral field and that

$$\bar{J}(\bar{z}) = \iota \partial_{\bar{z}} X$$

is an antichiral field. $J(z)$ and $\bar{J}(\bar{z})$ are quantities of particularly great significance, and their importance would become manifest as we discuss the Wess-Zumino-Witten (WZW) models in the next chapter, but for now, we say simply that their respective conformal dimensions could be worked out from the action if we assume that the field X is a primary field with vanishing conformal dimensions, something which, in light of Eq. (3.19), could mathematically be written as

$$X'(w, \bar{w}) = X(z, \bar{z}).$$

We do the following calculation:

$$\begin{aligned} S &= \frac{1}{2\pi\kappa} \int dw d\bar{w} \partial_w X'(w, \bar{w}) \partial_{\bar{w}} X'(w, \bar{w}) \\ &= \frac{1}{2\pi\kappa} \int \frac{dw}{dz} dz \frac{d\bar{w}}{d\bar{z}} d\bar{z} \frac{dz}{dw} \partial_z X(z, \bar{z}) \frac{d\bar{z}}{d\bar{w}} \partial_{\bar{z}} X(z, \bar{z}) \\ &= \frac{1}{2\pi\kappa} \int dz d\bar{z} \partial_z X(z, \bar{z}) \partial_{\bar{z}} X(z, \bar{z}). \end{aligned}$$

Not only does this calculation show that the action is invariant if X has vanishing conformal dimensions, but it also quite manifestly establishes that $J(z)$ and $\bar{J}(\bar{z})$ (which respectively appear as $\partial_z X(z, \bar{z})$ and $\partial_{\bar{z}} X(z, \bar{z})$ in the action here) have conformal dimensions given by $(1, 0)$ and $(0, 1)$, respectively.

Another important calculation could be done with the action, namely the calculation of the energy-momentum tensor, $T_{\alpha\beta}$. Two results would be used, which we state without proof:

$$T_{\alpha\beta} = 4\pi\kappa\gamma \frac{1}{\sqrt{|g|}} \frac{\partial S}{\partial g^{\alpha\beta}},$$

and

$$\delta\sqrt{|g|} = -\frac{1}{2}\sqrt{|g|}g_{\alpha\beta}\delta g^{\alpha\beta}.$$

Varying the action using them, we find that $T_{zz} = \gamma\partial_z X\partial_z X$, $T_{z\bar{z}} = T_{\bar{z}z} = 0$, and $T_{\bar{z}\bar{z}} = \gamma\partial_{\bar{z}} X\partial_{\bar{z}} X$. It must, however, be remembered that for a quantum theory, the ground-state expectation value of an energy-momentum tensor should be finite. Therefore, $T_{\alpha\beta}$ is written as a normal ordered expression:

$$T(z) = \gamma\mathcal{N}\left(\partial_z\hat{X}\partial_z\hat{X}\right)(z) = \gamma\mathcal{N}(\hat{j}\hat{j})(z). \tag{3.35}$$

Of course, we have given the result for the chiral part only. Each of the j's represents the chiral field $J(z)$.

3.4.6 Free fermionic field

The example to be discussed in this section is that of a free fermionic field in a 2-dimensional space, which we take as being the Minkowski space. The action of this field could be written as

$$S = \frac{1}{4\pi\kappa}\int dx^0 dx^1\sqrt{|h|}(-\iota)\bar{\Psi}\gamma^\alpha\partial_\alpha\Psi, \tag{3.36}$$

where $h = \det h_{\mu\nu}$ with

$$h_{\mu\nu} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

and κ is a normalization constant. Similar to what we dealt with in section 1.3, Ψ is a spinor, but it is a 2×1 column vector this time—rather than being 4×1. The γ^α, likewise, are matrices satisfying the Clifford algebra, but they are 2×2 this time around—rather than being 4×4. Also, there are only two of them now. If we see Eq. (3.36) closely, we notice that the action is almost the same as the one we wrote for the Dirac field (Eq. (1.13)), which is also a free fermionic field. The only major difference is that we do not have any mass term now, an attestation to the fact that we are dealing with a CFT.

Several representations of the γ^μ are possible, but we choose the following.

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} ; \gamma^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} .$$

Its choice has to do with the fact that under this representation, the components, ψ and $\bar{\psi}$, of the spinor Ψ are real. We then perform the Wick rotation $x_1 \rightarrow \imath x_1$, which effects the derivative transformation $\partial_1 \rightarrow -\imath\partial_1$. Essentially, this rotation is tantamount to making another choice for the matrices γ^α, namely

$$\gamma^0 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

and

$$\gamma^1 = \begin{bmatrix} 0 & -\imath \\ \imath & 0 \end{bmatrix} ,$$

and we see that it also introduces an additional factor of $\imath$ in the action, which comes from dx^1. We begin simplifying the action then, observing that $\bar{\Psi} = \Psi^\dagger \gamma^0$ and

$$\begin{aligned} &\gamma^0 \gamma^\mu \partial_\mu \\ &= \gamma^0 \left(\gamma^0 \partial_0 + \gamma^1 \partial_1\right) \\ &= \begin{bmatrix} \partial_0 + \imath\partial_1 & 0 \\ 0 & \partial_0 - \imath\partial_1 \end{bmatrix} \\ &= 2 \begin{bmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{bmatrix} . \end{aligned}$$

In the last step, we have again employed the change of variables in Eq. (3.14). This change allows us to write our spinor as

$$\Psi = \begin{bmatrix} \psi(z,\bar{z}) \\ \bar{\psi}(z,\bar{z}) \end{bmatrix},$$

where $\psi(z,\bar{z})$ and $\bar{\psi}(z,\bar{z})$ are still real fields. Using all this discussion, we could finally simplify the action as

$$\begin{aligned} S &= \frac{1}{4\pi\kappa}\int dz d\bar{z}\sqrt{|g|}2\Psi^\dagger \begin{bmatrix} \partial_{\bar{z}} & 0 \\ 0 & \partial_z \end{bmatrix}\Psi \\ &= \frac{1}{4\pi\kappa}\int dz d\bar{z}(\psi\partial_{\bar{z}}\psi + \bar{\psi}\partial_z\bar{\psi}). \end{aligned} \tag{3.37}$$

As to the metric, we used what we obtained from $h_{\mu\nu}$ after the coordinate change, namely

$$g_{\mu\nu} = \begin{bmatrix} 0 & 1/2 \\ 1/2 & 0 \end{bmatrix}.$$

Now, to do a CFT analysis, we could begin by varying the action with respect to ψ:

$$\begin{aligned} 0 &= \delta_\psi S \\ &= \frac{1}{4\pi\kappa}\int d^2z(\delta\psi\partial_{\bar{z}}\psi + \psi\partial_{\bar{z}}(\delta\psi)) \\ &= \frac{1}{4\pi\kappa}\int d^2z(\delta\psi\partial_{\bar{z}}\psi + \partial_{\bar{z}}(\psi\delta\psi) - (\partial_{\bar{z}}\psi)\delta\psi) \\ &= \frac{1}{2\pi\kappa}\int d^2z\delta\psi\partial_{\bar{z}}\psi. \end{aligned}$$

Since the variation in action has to be 0 for all $\delta\psi$, we have

$$\partial_{\bar{z}}\psi = 0. \tag{3.38}$$

A similar variation of the action with respect to $\bar{\psi}$ gives

$$\partial_z \bar{\psi} = 0 \tag{3.39}$$

What these equations indicate is that $J(z) = \psi$ is a chiral field whereas $\bar{J}(\bar{z}) = \bar{\psi}$ is an antichiral field. Next, we use the action to show that if $J(z)$ and $\bar{J}(\bar{z})$ are primary fields with conformal dimensions $(1/2, 0)$ and $(0, 1/2)$, respectively, then the action is invariant under conformal transformations:

$$\begin{aligned}
S &= \frac{1}{4\pi\kappa} \int dw d\bar{w} \left(\psi'(w,\bar{w}) \partial_{\bar{w}} \psi'(w,\bar{w}) + \bar{\psi}'(w,\bar{w}) \partial_w \bar{\psi}'(w,\bar{w}) \right) \\
&= \frac{1}{4\pi\kappa} \int \frac{dz}{dw} dw \frac{d\bar{z}}{d\bar{w}} d\bar{w} \left[\left(\frac{dz}{dw} \right)^{1/2} \psi(z,\bar{z}) \frac{d\bar{z}}{d\bar{w}} \partial_{\bar{z}} \left(\frac{dz}{dw} \right)^{1/2} \psi(z,\bar{z}) \right. \\
&\qquad \left. + \left(\frac{d\bar{z}}{d\bar{w}} \right)^{1/2} \bar{\psi}(z,\bar{z}) \frac{dz}{dw} \partial_z \left(\frac{d\bar{z}}{d\bar{w}} \right)^{1/2} \right] \\
&= \frac{1}{4\pi\kappa} \int dz d\bar{z} (\psi(z,\bar{z}) \partial_{\bar{z}} \psi(z,\bar{z}) + \bar{\psi}(z,\bar{z}) \partial_z \bar{\psi}(z,\bar{z})).
\end{aligned}$$

Clearly then, the action is invariant under conformal transformations if ψ and $\bar{\psi}$ are primary fields whose conformal dimensions are $(1/2, 0)$ and $(0, 1/2)$, respectively.

As in the example of the free bosonic field, we say that another important calculation involving the action is the calculation of the energy-momentum tensor, $T_{\mu\nu}$. For fermionic theories, the way $T_{\mu\nu}$ is defined differs from the way it is defined for bosonic theories, and we would have to introduce more structure before we could write its explicit form in the context of fermions. Therefore, we take a roundabout route and calculate it via an equivalent but different way:

$$T_{\mu\nu} = 8\pi\kappa\gamma \left(-g_{\mu\nu}\mathcal{L} + \sum_i \frac{\partial \mathcal{L}}{\partial(\partial^\mu \varphi_i)} \partial_\nu \varphi_i \right), \tag{3.40}$$

where γ is a normalization constant, $g_{\mu\nu}$ is the metric we found for complex coordinates, and the φ_i

stand for the fermionic fields. The energy-momentum tensor found via this equation is not symmetric in general, but it could be made symmetric with the equations of motion. For instance, we find that $T_{zz} = \gamma\psi\partial\psi$, $T_{z\bar{z}} = -\gamma\bar{\psi}\partial_z\bar{\psi}$, $T_{\bar{z}z} = -\gamma\psi\partial_{\bar{z}}\psi$, and $T_{\bar{z}\bar{z}} = \gamma\bar{\psi}\partial_{\bar{z}}\bar{\psi}$; whereas $T_{\mu\nu}$ is not symmetric so far, it could easily be made symmetric with the observation that ψ and $\bar{\psi}$ are chiral and antichiral fields, respectively, since this observation makes both $T_{z\bar{z}}$ and $T_{\bar{z}z}$ equal to 0.

We thus have obtained all the nonvanishing components of the energy-momentum tensor for our fermionic field. Using normal ordering as we did earlier and stating the result for the chiral part only, we could write

$$T(z) = \gamma\mathcal{N}\left(\hat{\psi}\partial_z\hat{\psi}\right) . \tag{3.41}$$

"You enter a completely new world where things aren't at all what you're used to."

Edward Witten

4

Wess-Zumino-Witten models[2,3,4]

Marking the ultimate culmination of this work, this final chapter uses whatever we have covered up till now (about quantum fields in general and CFTs in particular) to first discuss the famous Wess-Zumino-Witten models and then show that they comprise CFTs in 2 dimensions. In doing so, it goes through the so-called Sugawara construction of the energy-momentum tensor.

The general scheme of things starts with a short discourse on nonlinear sigma models and identifies what prevents them from being CFTs. Then, Edward Witten's contribution to the action of a nonlinear sigma model is addressed, and it is shown that the term he introduces does hint at the fact that the model has turned into a CFT. The current algebra of the new model is then presented as pointing toward an in-built Sugawara construction, and this observation is verified together with the fact that the Wess-Zumino-Witten model is indeed a CFT.

4.1 Nonlinear sigma models

We begin our discussion with the nonlinear sigma model, whose action could be written as

$$S_0 = \frac{1}{4\lambda^2}\int dz d\bar{z} \mathrm{Tr}(\partial^\mu g^{-1}\partial_\mu g). \tag{4.1}$$

λ here is a dimensionless coupling constant, and g could be viewed as a map from the Riemann sphere to the Lie algebra G, that is, $g : S^2 \to G$. The trace here represents an invariant product for the relevant Lie algebra, and it is normalized such that $\mathrm{Tr}\left(T^a T^b\right) = (1/2)\delta^{ab}$ for any two generators of the Lie algebra, T^a and T^b. That this action does not give a CFT could be seen by some comparisons with the examples of the free bosonic and fermionic fields that we covered in chapter 3. Varying the action, for instance, we see the following:

$$\begin{aligned}
0 &= \delta S_0 \\
&= \frac{1}{2\lambda^2}\int dz d\bar{z} \mathrm{Tr}[(-g^{-1}\delta g g^{-1}\partial_\mu g + g^{-1}\delta(\partial_\mu g))g^{-1}\partial^\mu g] \\
&= \frac{1}{2\lambda^2}\int dz d\bar{z} \mathrm{Tr}(\partial_\mu(g^{-1}\delta g)g^{-1}\partial^\mu g) \\
&= -\frac{1}{2\lambda^2}\int dz d\bar{z} \mathrm{Tr}(g^{-1}\delta g \partial_\mu(g^{-1}\partial^\mu g)),
\end{aligned}$$

where we used the cyclic invariance of the trace in the second step and integration by parts in the third. Since we want the variation of the action to be 0 for all δg, we could say that the equation of motion is

$$\partial_\mu(g^{-1}\partial^\mu g) = 0. \tag{4.2}$$

What this equation says is that $J^\mu = g^{-1}\partial^\mu g$ is conserved. If we try, however, to construct the chiral and antichiral J's as we did in sections 3.4.5 and 3.4.6, we see that we badly fail. If, for instance, $\tilde{J}_z = g^{-1}\partial_z g$ and $\tilde{J}_{\bar{z}} = g^{-1}\partial_{\bar{z}} g$, we find that

$$\partial_z \tilde{J}_{\bar{z}} + \partial_{\bar{z}} \tilde{J}_z = 0,$$

but it is particularly easy to check that neither of these terms is necessarily equal to 0 on its own. In fact, their being equal to 0 simultaneously leads to an inconsistency in that $\partial_z(g^{-1}\partial_{\bar{z}} g) = 0$ requires

$$\partial_z \partial_{\bar{z}} g = \partial_{\bar{z}} g g^{-1} \partial_z g.$$

Then, since the partial derivatives on the left-hand side could conveniently be interchanged, we may write that

$$\partial_{\bar{z}} g g^{-1} \partial_z g = \partial_z g g^{-1} \partial_{\bar{z}} g.$$

This statement is an equality of the form $abc = cba$, which is not expected to hold in general for non-Abelian algebras. The correct choice for the J's is given by $J_z = \partial_z g g^{-1}$ and $J_{\bar{z}} = g^{-1}\partial_{\bar{z}} g$, and if $\partial_{\bar{z}} J_z$ is equal to 0, $\partial_z J_{\bar{z}}$'s being equal to 0 is implied necessarily:

$$\partial_z(g^{-1}\partial_{\bar{z}} g) = g^{-1}\partial_{\bar{z}}(\partial_z g g^{-1}) g. \tag{4.3}$$

However, neither of these J's is either chiral or antichiral in the nonlinear sigma model, and this thing hints at the fact that unlike the free bosonic and fermionic theories discussed in chapter 3, the nonlinear sigma model is not a CFT.

4.2 Wess-Zumino-Witten models

In order to turn the nonlinear sigma model into a CFT, Edward Witten suggested the addition of the term

$$\Gamma = \frac{-\iota}{24\pi} \int d^3y \varepsilon_{\alpha\beta\gamma} \mathrm{Tr} \left(\tilde{g}^{-1} \partial^\alpha \tilde{g} \tilde{g}^{-1} \partial^\beta \tilde{g} \tilde{g}^{-1} \partial^\gamma \tilde{g} \right) \tag{4.4}$$

to the action in Eq. (4.1). Γ is defined on a 3-dimensional region, and $\tilde{g}$ represents the extension of g from the Riemann sphere into this region. Since this extension is not unique, there is an ambiguity in the definition of Γ. This ambiguity is quantifiable, but we would leave its discussion here, for it is not germane to the task at hand; we just make the comment that the ambiguity is defined modulo $2\pi\iota$.

The action we consider now is then $S = S_0 + k\Gamma$, where S_0 is given by Eq. (4.1) and k is just some coupling constant. Whereas S is a valid action classically for any value of k, quantum theory restricts the values that k is allowed to take. Since the ambiguity in Γ is defined modulo $2\pi\iota$, path integrals in complex time,

$$\int [d\varphi] \exp(-S_0) \exp(-k\Gamma),$$

would be well-defined only if $k \in \mathbb{Z}$. Even though Γ is a 3-dimensional integral, variation in it due to $g \to g + \delta g$ is a 2-dimensional one, for the variation could be written as a total derivative:

$$\int d^3y \varepsilon_{\alpha\beta\gamma} \partial^\gamma (\ldots) = \int dz d\bar{z} \varepsilon_{\alpha\beta} (\ldots).$$

The final result of the variation is

$$\delta\Gamma = \frac{\iota}{8\pi} \int d^2x \varepsilon_{\mu\nu} \mathrm{Tr}(g^{-1} \partial g \partial^\mu (g^{-1} \partial^\nu g)), \tag{4.5}$$

and the equation of motion obtained from the variation of the full action S (in complex coordinates) is

$$\left(1+\frac{\lambda^2 k}{4\pi}\right)\partial_z\left(g^{-1}\partial_{\bar{z}}g\right)+\left(1-\frac{\lambda^2 k}{4\pi}\right)\partial_{\bar{z}}\left(g^{-1}\partial_z g\right)=0. \tag{4.6}$$

Now, if λ^2 is set equal to $4\pi/k$, the equation of motion becomes $\partial_z(g^{-1}\partial_{\bar{z}}g) = 0$, which tells us that $J_{\bar{z}} = g^{-1}\partial_{\bar{z}}g$ is antichiral. In view of Eq. (4.3), it also tells us that $\partial_z g g^{-1}$ is chiral. On the other hand, if λ^2 is set equal to $-4\pi/k$, then the equation of motion turns into $\partial_{\bar{z}}(g^{-1}\partial_z g) = 0$, meaning that $J_z = g^{-1}\partial_z g$ is chiral and, in the light of Eq. (4.3), $J_{\bar{z}} = \partial_{\bar{z}} g g^{-1}$ is antichiral. The former choice for λ^2 defines the *Wess-Zumino-Witten* (WZW) *model*, whose complete action could be written as

$$S=\frac{k}{16\pi}\int dz d\bar{z}\,\mathrm{Tr}(\partial^\mu g^{-1}\partial_\mu g)+k\Gamma. \tag{4.7}$$

This choice, however, is irrelevant; what is important is fact that $\partial_z J_{\bar{z}}$ and $\partial_{\bar{z}} J_z$ are now separately 0. This very thing hints at that the WZW model is a 2-dimensional CFT, and we go on verify this hunch of ours in the subsequent sections.

4.3 Current algebras

This section is meant to introduce the notions of *currents* and *current algebras* in 2-dimensional CFTs. Beginning by expounding what we mean by currents, we proceed to presenting a couple of ways to reach the kind of algebra they form. All these concepts would prove particularly useful in the subsequent section, for they would help us establish that a WZW model is a 2-dimensional CFT.

A chiral primary field, $J(z)$, with a conformal dimension of 1 is referred to as a current, and a similar definition holds for an antichiral field as well. If a theory has N such fields, we could, as per our discussion in the previous chapter, Laurent expand each of them like

$$J^i(z)=\sum_{n\in\mathbb{Z}} z^{-n-1}J^i_n$$

and determine the algebra that the Laurent modes J_n^c would satisfy. Doing so requires us to resort to the general expression for the commutation relation of two chiral primary fields and use a conformal dimension of 1 for each. Some simplifications too are required, but we leave all these things for now in the interest of covering what is essential and simply quote the result:

$$\left[J_m^a, J_n^b\right] = \imath \sum_c f^{ab}_{\ c} J^c_{m+n} + km\delta^{ab}\delta_{m+n,0}. \tag{4.8}$$

The algebra thus obtained is a generalization of a Lie algebra and happens to be infinite-dimensional (just like the Witt and the Virasoro algebras). Known as the so-called *Kac-Moody algebra*, this algebra could also be arrived at via an approach that uses OPEs. As shown in section 3.4.1, OPEs are closely related to commutation relations and, hence, to algebras. We could go about determining the OPE of some current in a 2-dimensional CFT with another current in the theory through arguments about the OPE's possible structure.

Starting with the holomorphic current $J(z) \sim \partial_z g g^{-1}$ in a WZW model, for instance, we could use dimensional analysis to see that it has conformal dimension 1. Thus, when writing the OPE in the form

$$J^a(z)J^b(w) \sim \sum_p \frac{X_p(w)}{(z-w)^p},$$

we realize that the holomorphic field $X_p(w)$ should have conformal dimension $2-p$. Unitarity also puts constraints on the possible form of the OPE; for example, it stipulates that there be no operators with negative conformal dimensions. The highest allowed pole order for our OPE thus is 2. A field with conformal dimension 0 being proportional to identity and a field with conformal dimension 1 being a current itself, we could finally go on to say that our OPE has the form

$$R\left(J^a(z)J^b(w)\right) \sim \frac{\kappa^{ab}}{(z-w)^2} + \frac{\imath f^{ab}_{\ c} J^c(w)}{z-w},$$

where κ^{ab} and $f^{ab}_{\ c}$ are some constants and we have dropped the nonsingular terms for now. Symmetry under the exchange of the currents and the associativity of the OPEs tell us that $\kappa_{ab} = \kappa_{ba}$ and that the $f^{ab}_{\ c}$ are the structure constants of a Lie algebra. We could choose a basis for the relevant Lie algebra such that $\kappa^{ab} = k\delta^{ab}$, and we identify this k with the level of a WZW model. A refined expression for the OPE at hand could hence be written as

$$R\left(J^a(z)J^b(w)\right) \sim \frac{k\delta^{ab}}{(z-w)^2} + \frac{\imath f^{ab}_{\ c} J^c(w)}{z-w}, \tag{4.9}$$

where we have dropped the nonsingular terms again. Since this current OPE is closely related to the corresponding current algebra, it is sometimes referred to as the current algebra itself. In fact, it could be used to work out the related current algebra. Laurent expanding the currents as

$$J^a(z) = \sum_{n\in\mathbb{Z}} J^a_n z^{-n-1},$$

extracting the modes with contour integration as

$$J^a_n = \oint dz z^n J^a(z),$$

and putting their expressions into a commutator yield

$$\begin{aligned}
\left[J^a_m, J^b_n\right] &= \oint_{C(0)} \frac{dw}{2\pi\imath} \oint_{C(w)} \frac{dz}{2\pi\imath} z^m w^n R\left(J^a(z)J^b(w)\right) \\
&= \oint_{C(0)} \frac{dw}{2\pi\imath} \oint_{C(w)} \frac{dz}{2\pi\imath} z^m w^n \left(\frac{k\delta^{ab}}{(z-w)^2} + \frac{\imath f^{ab}_{\ c} J^c(z)}{z-w}\right) \\
&= \oint_{C(0)} \frac{dw}{2\pi\imath} \left(km\delta^{ab} w^{m+n-1} + \imath f^{ab}_{\ c} J^c(z) w^{m+n}\right) \\
&= km\delta^{ab}\delta_{m+n,0} + \imath f^{ab}_{\ c} J^c_{m+n}.
\end{aligned}$$

Clearly then, the currents of the WZW model too satisfy the Kac-Moody algebra, which we discussed above. The modes with $m = n = 0$ satisfy a Lie alegbra; the extra term $km\delta^{ab}\delta_{m+n,0}$ represents a central extension of kind that Witt algebra was shown to admit. The fact that the currents of the WZW model could be seen as the so-called *Kac-Moody currents* would turn out to be of great significance in the next section and would help us prove that this model is a CFT.

4.4 Sugawara construction

In chapter 3, we saw that 2-dimensional CFTs are characterized by the Virasoro algebra, which was the central extension of the Witt algebra and was generated by the energy-momentum tensor. If we now consider some other algebras characteristic of a 2-dimensional CFT, then they would have to be compatible with the Virasoro algebra. This fact implies the existence of an inherent definition of the energy-momentum tensor. Looking at the current algebras, for instance, we should find that this inherent definition is such that each current has conformal dimension 1 with respect to the chiral part of the energy-momentum tensor. In fact, the energy-momentum tensor itself should satisfy the Virasoro algebra, something which, in light of the discussion in section 3.4.2, implies that

$$R\left(\hat{T}(z)\hat{T}(w)\right) = \frac{c/2}{(z-w)^4} + \frac{2\hat{T}(w)}{(z-w)^2} + \frac{\partial_w \hat{T}(w)}{z-w} + \dots .$$

Put a little differently, if a theory is conformal, then its energy-momentum tensor should yield the OPE above.

Guided by the example of the free bosonic field in section 3.4.5, we investigate the following ansatz for the chiral part, $T(z)$, of the energy-momentum tensor of a CFT:

$$T(z) = \gamma \sum_{a=1}^{\dim \mathfrak{g}} \mathcal{N}\left(J^a J^a\right)(z), \tag{4.10}$$

where $\mathfrak{g}$ represents the relavant Lie algebra. If we now require that each of the J^a be a primary field of conformal dimension 1, we could use the normal ordering result in Eq. (3.31) to write our ansatz in terms of the Laurent modes of the energy-momentum tensor:

$$L_m = \gamma \sum_{a=1}^{\dim \mathfrak{g}} \left(\sum_{l \leq -1} J^a_l J^a_{m-l} + \sum_{l > -1} J^a_{m-l} J^a_l \right).$$

Recalling another important result from the previous chapter, namely the one at the end of section 3.4.3, we would do some calculations that would help us fix γ.

$$\begin{aligned}
[L_m, J^a_n] &= \gamma \sum_b \left(\sum_{l \leq -1} \left[J^b_l J^b_{m-l}, J^a_n \right] + \sum_{l > -1} \left[J^b_{m-l} J^b_l, J^a_n \right] \right) \\
&= -2\gamma n k J^a_{m+n} + \gamma \sum_{bc} i f^b_{ac} \sum_{l \leq -1} \left(J^b_l J^c_{m+n-l} + J^c_{l+n} J^b_{m-l} \right) \\
&\quad + \gamma \sum_{bc} i f^{ba}_{c} \sum_{l > -1} \left(J^c_{m+n-l} J^b_l + J^b_{m-l} J^c_{l+n} \right),
\end{aligned}$$

where we assumed that the current Laurent modes satisfy the Kac-Moody algebra. Now, since $f^{ba}_{\ \ c} = -f^{ca}_{\ \ b}$, we could further simply the final expression we got in the following manner:

$$\begin{aligned}
[L_m, J^a_n] &= -2\gamma n k J^a_{m+n} - \gamma \sum_{bc} i f^{ba}_{\ \ c} \sum_{l=0}^{n-1} \left[J^b_l, J^c_{m+n-l} \right] \\
&= -2\gamma n k J^a_{m+n} - \gamma \sum_{bc} i f^{ba}_{\ \ c} \sum_{l=0}^{n-1} \sum_d i f^{bc}_{\ \ d} J^d_{m+n} \\
&= -2\gamma n k J^a_{m+n} + \gamma n \sum_{bcd} f^{ba}_{\ \ c} f^{bc}_{\ \ d} J^d_{m+n}.
\end{aligned}$$

We know that the structure constants satisfy $\sum_{bc} f^{ba}_{\ \ c} f^{bc}_{\ \ d} = -2C_{\mathfrak{g}} \delta^{ab}$, where $C_{\mathfrak{g}}$ is the dual Coxeter number of the Lie algebra $\mathfrak{d}$, so

$$[L_m, J^a_n] = -2\gamma n\,(k + C_{\mathfrak{d}}) J^a_{m+n}.$$

Finally, knowing that each of the J^a has conformal dimension 1 and comparing the final result above with the aforementioned result from section 3.4.3, we find that $\gamma^{-1} = 2(k + C_{\mathfrak{d}})$ and, hence, that

$$T(z) = \frac{1}{2(k + C_{\mathfrak{d}})} \sum_{a=1}^{\dim\,\mathfrak{d}} \mathcal{N}\left(J^a J^a\right)(z). \tag{4.11}$$

What we have now is the so-called *Sugawara constructed* energy-momentum tensor of a CFT in 2 dimensions, and its construction is what is known as the *Sugawara construction*. It is this very construction that we would use to show that the a WZW model comprises a 2-dimensional CFT; however, before going down that route, we state without proof that the central charge of a CFT characterized by Kac-Moody currents with respect to the Sugawara constructed energy-momentum tensor is

$$c = \frac{k \dim\,\mathfrak{d}}{k + C_{\mathfrak{d}}}, \tag{4.12}$$

a result which we would use to further corroborate our conclusion that the WZW model describes a 2-dimensional CFT.

We begin with an observation we made in the previous section, namely that the currents of a WZW model satisfy the Kac-Moody algebra. Based on our discussion in this section then, we could easily say that such models have an in-built Sugawara construction. We just need to show that such is indeed the case. Beginning by identifying the energy-momentum tensor of the model, we see that it gives

$$T(z) = \gamma \mathcal{N}\left(J^a J^a\right)(z),$$

where normal ordering, as expressed in Eq. (3.30), could be thought of as the nonsingular part of the OPE yielded by the J's. This nonsingular part could quite conveniently be extracted via contour integration in the following manner:

$$\mathcal{N}(J^a J^a)(z) = \frac{1}{2\pi\imath} \oint_{C(z)} \frac{dw}{w-z} J^a(w) J^a(z).$$

With this result, we could go and calculate the OPE of $J^a(z)$ with $T(w)$. We are interested only in the singular part of this OPE, which we denote by a line like the one we used for Wick contractions:

$$\begin{aligned}
J^a(z)T(w) &= \frac{\gamma}{2\pi\imath} \oint_{C(w)} \frac{dx}{x-w} J^a(z)\Big(J^b(x)J^b(w)\Big) \\
&= \frac{\gamma}{2\pi\imath} \oint_{C(w)} \frac{dx}{x-w} \left(\Big(J^a(z)J^b(x)\Big)J^b(w) + J^b(x)\Big(J^a(z)J^b(w)\Big)\right) \\
&= \frac{\gamma}{2\pi\imath} \oint_{C(w)} \frac{dx}{x-w} \left[\left(\frac{k\delta^{ab}}{(z-x)^2} + \frac{\imath f^{ab}_{\ \ c} J^c(x)}{z-x}\right)J^b(w)\right. \\
&\qquad\qquad\qquad\qquad \left. + J^b(x)\left(\frac{k\delta^{ab}}{(z-w)^2} + \frac{\imath f^{ab}_{\ \ c} J^c(w)}{z-w}\right)\right].
\end{aligned}$$

Now, we need to calculate the remaining JJ OPEs. Simplifying the final expression we just obtained, we observe that since the structure constants are totally antisymmetric and the Kronecker delta is symmetric, the second order poles in the OPEs do not contribute, so

$$
\begin{aligned}
J^a(z)T(w) = \frac{\gamma}{2\pi\iota}\oint_{C(w)} \frac{dx}{x-w}\Bigg[&\frac{k\delta^{ab}J^b(w)}{(z-x)^2} + \frac{k\delta^{ab}J^b(x)}{(z-w)^2} \\
&+ \frac{\iota f^{ab}{}_c}{z-x}\left(\frac{\iota f^{cb}{}_d J^d(w)}{x-w} + \mathcal{N}(J^c J^b)(w)\right) \\
&+ \frac{\iota f^{ab}{}_c}{x-w}\left(\frac{\iota f^{bc}{}_d J^d(w)}{x-w} + \mathcal{N}(J^c J^b)(w)\right)\Bigg] \\
= \gamma\left(\frac{2k\delta^{ab}J^b(w)}{(w-z)^2} - \frac{f^{ab}{}_c f^{cb}{}_d J^d(w)}{(w-z)^2}\right)&,
\end{aligned}
$$

where we obtained the last line by noticing that the normal ordered terms vanish after integration: they cancel due to the antisymmetry in b and c. Finally, we recall the definition of the dual Coxeter number—$f^{ab}{}_c f^{bc}{}_d = 2C_\eth \delta^{ad}$—thereby obtaining

$$
\begin{aligned}
T(z)J^a(w) &= J^a(z)T(w) \\
&= 2\gamma(k + C_\eth)\frac{J^a(z)}{(w-z)^2} \\
&= 2\gamma(k + C_\eth)\left(\frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{z-w}\right).
\end{aligned}
$$

In order for $J^a(w)$ to have a conformal dimension of 1, we need to set γ the way we did before: $\gamma^{-1} = 2(k + C_\eth)$. Once γ has been fixed this way, we get what should indeed be the OPE of a primary field of conformal dimension 1 with the chiral part of the energy momentum tensor:

$$
T(z)J^a(w) = \frac{J^a(w)}{(z-w)^2} + \frac{\partial_w J^a(w)}{z-w} + \dots,
$$

where ... represent the nonsingular terms.

What now needs to be shown is that the OPE $R(T(z)T(w))$ is such that it satisfies the Virasoro algebra. This thing would establish that the WZW models indeed are CFTs. We are particularly interested in seeing what the central charge turns out to be as well, for its value would further corroborate what we stated earlier: the WZW model has an in-built Sugawara construction.

$$
\begin{aligned}
\overline{T(z)T}(w) &= \frac{1}{4\pi\imath(k+C_{\eth})}\oint\frac{dx}{x-w}((\overline{T(z)J^a}(x))J^a(w)+J^a(x)(\overline{T(z)J^a}(w))) \\
&= \frac{1}{4\pi\imath(k+C_{\eth})}\oint\frac{dx}{x-w}\left(\frac{k\dim\eth}{(z-x)^2(x-w)^2}-\frac{2k\dim\eth}{(z-x)(x-w)^3}\right. \\
&\qquad\left.+\frac{k\dim\ \eth}{(x-w)^2(z-w)^2}+\frac{2k\dim\eth}{(z-w)(x-w)^3}\right) \\
&\quad+\frac{1}{4\pi\imath(k+C_{\eth})}\oint\frac{dx}{x-w}\left(\frac{\mathcal{N}(J^aJ^a)(w)}{(z-x)^2}+\frac{\mathcal{N}(\partial_wJ^aJ^a)(w)}{z-x}\right. \\
&\qquad\left.+\frac{\mathcal{N}(J^aJ^a)(w)}{(z-w)^2}+\frac{\mathcal{N}(J^a\partial_wJ^a)(w)}{z-w}\right) \\
&= \frac{(3-2+0+0)k\dim\eth}{2(k+C_{\eth})(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_wT(w)}{z-w} \\
&= \frac{k\dim\eth}{2(k+C_{\eth})(z-w)^4}+\frac{2T(w)}{(z-w)^2}+\frac{\partial_wT(w)}{z-w},
\end{aligned}
$$

which is definitely the OPE we were looking for. Not only does it show that the energy-momentum tensor satisfies the Virasoro algebra, but it also demonstrates that the central charge is $k\dim\eth/(k+C_{\eth})$, which indicates that the energy-momentum tensor in a WZW model is Sugawara constructed.

5

Conclusion

Now that we have attained all the goals that set out in the work to accomplish, we would briefly summarize them here one by one.

In chapter 0, not only did we discuss what a quantum field means, but we also established our motivations for considering such a contraption. Bringing locality in was definitely one, but we saw that classical fields could already do so pretty well. Consequently, we resorted to describing quantum fields as entities that span all of spacetime and equip us with the ability to create and destroy particles at any point therein. We also viewed quantum fields as explaining the indistinguishability of particles in the universe. Of course, if a single entity is defined everywhere, it must be capable of creating particles of the same constitution at all places.

Proceeding to make these ideas precise in chapter 1, we established a quantum field as the quantized version of a classical field through two major schemes of quantization: canonical quantization and path integral formalism. To demonstrate the former concretely, we showed its full application to the Klein-Gordon (KG) and Dirac fields. Toward the end of the chapter then, when we introduced the path integral formalism, it became easier to see the greater facility it affords in comparison with the program of canonical quantization. For one thing, path integrals use classical quantities, which are easier to deal with than operators, states, etc., to compute quantum amplitudes.

With the mathematics of quantum field theory in hand, we used chapter 2 to compute some physically important quantities. We chose to calculate correlators, for they would later help us transition into the realm of conformal field theory, the quantum field theory that this work was supposed to be particularly focused on. Calculating 2-point correlators for the KG and Dirac fields via both canonical quantization and path integral formalism, we were again able to see the greater ease that path integrals come with.

Then, in chapter 3, we started doing conformal field theory, discussing conformal symmteries in the context of both classical and quantum field theories. Doing so allowed us to observe how easy conformal symmetries render the calculation of quantities like correlators. Then, having investigated the relatively simple cases of conformal field theories in $d = 1$ and $d \geq 3$ dimensions, we began considering the exceedingly rich case of $d = 2$ dimensions and presenting a detailed discourse on such important topics as primary fields and operator product expansions. Finally, as concrete examples of conformal field theories in 2 dimensions, we presented the free, massless bosonic and fermionic fields on an infinite cylinder. Not only did these examples help us see the use of the conformal field theory jargon we had been developing in the context of some applications, but they also allowed us to get well-versed with the way conformal field theory techniques had to be applied.

Finally, in chapter 4, we began using our conformal field theory knowledge to discuss the famous Wess-Zumino-Witten (WZW) models. Considering the potential of a nonlinear sigma model to be a conformal field theory, we identified a problem with it and saw how it was fixed by a contribution of Edward Witten's. The new model, which is known as the WZW model, did seem to be a conformal field theory, but we still went on to verify this thing by checking the model's compatibility with the Virasoro algebra through the so-called Sugawara construction. Like our scrutiny of the cases of free, massless bosonic and fermionic fields, this exercise demonstrated the application of some pretty important conformal field theory techniques, such as the ways to look for compatibility between various algebras, methods involved in dealing with importanr operator product expansions, and so on.

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